A priori and *a posteriori* error identities for convex minimization problems based on convex duality relations

Lecture 4

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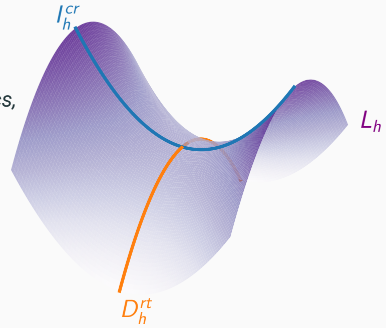
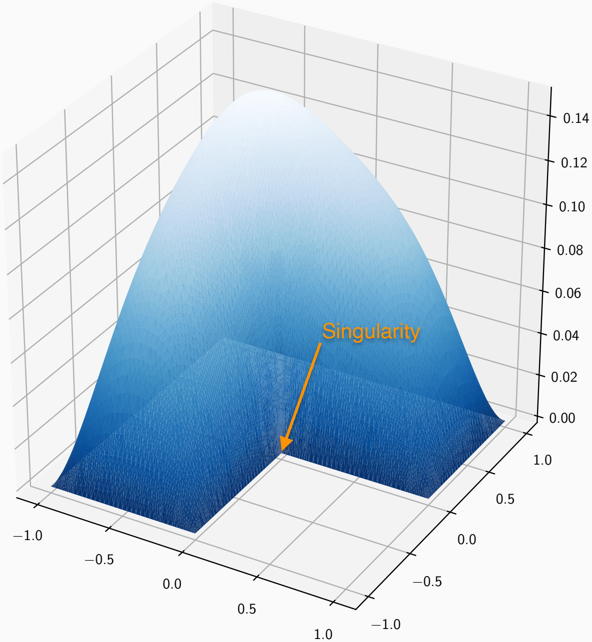


Content of the Lecture 4

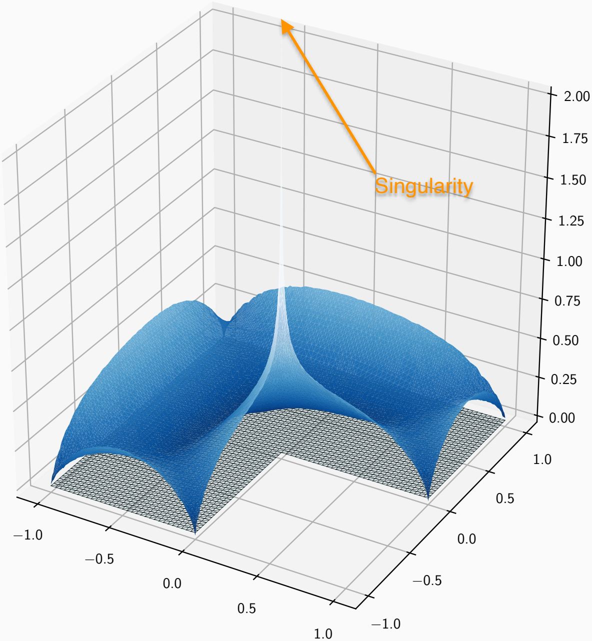
- **Lecture 4: Applications**
  - Adaptive mesh-refinement
    - Adaptive algorithm;
    - Examples.
  - *A priori* error identity on the basis of discrete convex duality;
    - Discrete primal-dual gap identity;
    - Examples.

Adaptive mesh-refinement

Adaptive mesh-refinement



Adaptive mesh-refinement



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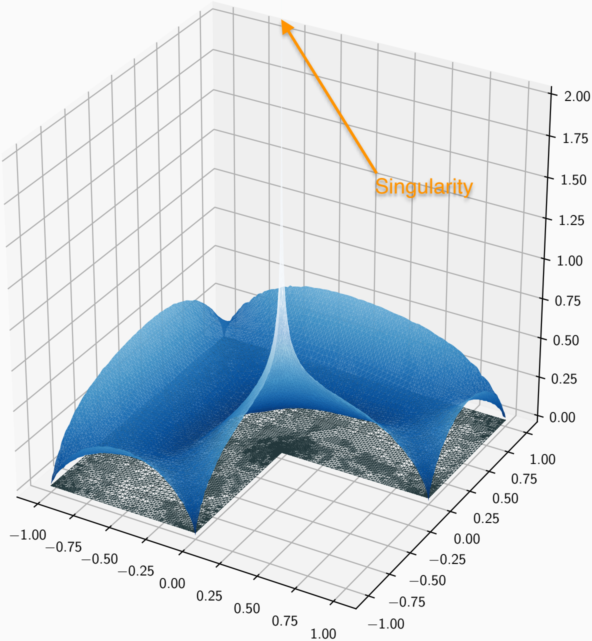

Adaptive algorithm

Algorithm: (SOLVE–ESTIMATE–MARK–REFINE)

Let $\varepsilon_{\mathrm{STOP}} > 0$, $\theta \in (0,1]$, and $\mathcal{T}_0$ an initial triangulation. Then, for every $k \geq 0$:

('SOLVE') Compute discrete primal solution and discrete dual solution

$$u_k^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k) \qquad \& \qquad z_k^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_k)\,;$$

('ESTIMATE') Post-process $u_k^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k)$ and $z_k^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_k)$ to obtain

$$\overline{u}_k^{cr} \in \mathrm{dom}(I) \qquad \& \qquad \overline{z}_k^{rt} \in \mathrm{dom}(-D) \cap W_N^{p'}(\mathrm{div};\Omega)\,;$$

Compute *(local) refinement indicators* $\{\eta^2_{\mathrm{gap},T}(\overline{u}_k^{cr}, \overline{z}_k^{rt})\}_{T\in\mathcal{T}_k}$, defined by

$$\eta^2_{\mathrm{gap},T}(\overline{u}_k^{cr}, \overline{z}_k^{rt}) := \left\{\begin{aligned} &\int_T \Big\{\phi^*(\cdot,\overline{z}_k^{rt}) - \overline{z}_k^{rt}\cdot\nabla\overline{u}_k^{cr} + \phi(\cdot,\nabla\overline{u}_k^{cr})\Big\}\,\mathrm{d}x \\ &\int_T \Big\{\psi^*(\cdot,\mathrm{div}\,\overline{z}_k^{rt}) - \mathrm{div}\,\overline{z}_k^{rt}\,\overline{u}_k^{cr} + \psi(\cdot,\overline{u}_k^{cr})\Big\}\,\mathrm{d}x \end{aligned}\right\}$$

for all $T \in \mathcal{T}_k$. If $\rho^2_{\mathrm{tot}}(\overline{u}_k^{cr}, \overline{z}_k^{rt}) = \eta^2_{\mathrm{gap}}(\overline{u}_k^{cr}, \overline{z}_k^{rt}) \leq \varepsilon_{\mathrm{STOP}}$, then STOP;

('MARK') Use *marking strategy* to find $\mathcal{M}_k \subseteq \mathcal{T}_k$ on which error is *'concentrated'*;

('REFINE') Perform *'refinement'* of $\mathcal{T}_k$ to get $\mathcal{T}_{k+1}$ s.t. each $T \in \mathcal{M}_k$ is *'refined'* in $\mathcal{T}_{k+1}$.

Increase $k \to k+1$ and continue with ('SOLVE').

- **Two Cases:**
  - **Case I:** Primal problem can be approximated cheaper:
    - → Compute $u_k^{cr} \in \mathcal{S}^{1,cr}_D(\mathcal{T}_k)$ using problem-dependent method;
    - → Compute $z_k^{rt} \in \mathcal{RT}^0_N(\mathcal{T}_k)$ via discrete reconstruction formula (for free!), *e.g.*,

$$z_k^{rt} = D_t\phi_{h_k}(\cdot, \nabla_{h_k} u_k^{cr}) + \frac{D_t\psi_{h_k}(\cdot, \Pi_{h_k} u_k^{cr})}{d}(\mathrm{id}_{\mathbb{R}^d} - \Pi_{h_k}\mathrm{id}_{\mathbb{R}^d})\,;$$

    - → **Examples:** Poisson problem, p-Dirichlet problem, Obstacle problem, ...
  - **Case II:** Dual problem can be approximated cheaper:
    - → Compute $z_k^{rt} \in \mathcal{RT}^0_N(\mathcal{T}_k)$ using problem-dependent method;
    - → Compute $u_k^{cr} \in \mathcal{S}^{1,cr}_D(\mathcal{T}_k)$ via discrete reconstruction formula (for free!), *e.g.*,

$$u_k^{cr} = D_t\psi^*_{h_k}(\cdot, \operatorname{div} z_k^{rt}) + D_t\phi^*_{h_k}(\cdot, \Pi_{h_k} z_k^{rt}) \cdot (\mathrm{id}_{\mathbb{R}^d} - \Pi_{h_k}\mathrm{id}_{\mathbb{R}^d})\,;$$

    - → **Example:** Elasto-plastic torsion problem, *i.e.*, $\phi := \frac{1}{2}|\cdot|^2 + I^{\Omega}_{|\cdot|\le 1} \notin C^0(\mathbb{R}^d)$ and $\psi(x,\cdot) := (t \mapsto -f(x)t) \in C^1(\mathbb{R})$ for a.e. $x \in \Omega$.

Adaptive algorithm

Algorithm: (SOLVE–ESTIMATE–MARK–REFINE)

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$$u_k^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k) \qquad \& \qquad z_k^{rt} \in \mathcal{R}T_N^0(\mathcal{T}_k)\,;$$

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$$\eta^2_{\mathrm{gap},T}(\overline{u}_k^{cr},\overline{z}_k^{rt}) \coloneqq \left\{\begin{aligned} &\int_T \Big\{\phi^*(\cdot,\overline{z}_k^{rt}) - \overline{z}_k^{rt}\cdot\nabla\overline{u}_k^{cr} + \phi(\cdot,\nabla\overline{u}_k^{cr})\Big\}\,\mathrm{d}x \\ &\int_T \Big\{\psi^*(\cdot,\mathrm{div}\,\overline{z}_k^{rt}) - \mathrm{div}\,\overline{z}_k^{rt}\,\overline{u}_k^{cr} + \psi(\cdot,\overline{u}_k^{cr})\Big\}\,\mathrm{d}x \end{aligned}\right.$$

for all $T \in \mathcal{T}_k$. If $\eta^2_{\mathrm{gap}}(\overline{u}_k^{cr},\overline{z}_k^{rt}) \leq \varepsilon_{\mathrm{STOP}}$, then STOP;

('MARK') Use *marking strategy* to find $\mathcal{M}_k \subseteq \mathcal{T}_k$ on which error is *'concentrated'*;

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Increase $k \to k+1$ and continue with ('SOLVE').

- **Non-conformity:**
  - Since
$$u_k^{cr} \in \mathcal{S}^{1,cr}_D(\mathcal{T}_k) \not\subseteq W^{1,p}_D(\Omega)\,,$$
in general, we have that
$$u_k^{cr} \notin \mathrm{dom}(I)\,.$$
  - Even if
$$z_k^{rt} \in \mathcal{R}T^0_N(\mathcal{T}_k) \subseteq W^{p'}_N(\mathrm{div};\Omega)\,,$$
in general, we have that
$$z_k^{rt} \notin \mathrm{dom}(-D)\,.$$

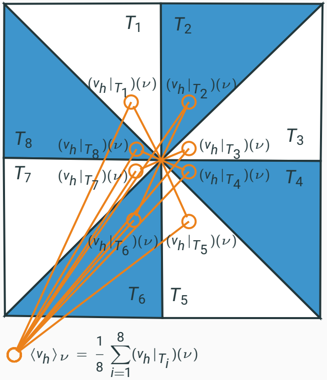
- **Post-processing:** Obtaining
$$\overline{u}_k^{cr} \in \mathrm{dom}(I) \qquad \& \qquad \overline{z}_k^{rt} \in \mathrm{dom}(-D) \cap W^{p'}_N(\mathrm{div};\Omega)\,,$$
is problem-dependent and often involves:
  - Quasi-interpolation operators (*e.g.*, node-averaging operator);
  - Truncation and scaling arguments that preserve regularity.

- **Node-averaging operator:** $\Pi_h^{av}\colon \mathbb{P}^1(\mathcal{T}_h) \to \mathcal{S}_D^1(\mathcal{T}_h)$, for every $v_h \in \mathbb{P}^1(\mathcal{T}_h)$ defined by

$$\Pi_h^{av} v_h \coloneqq \sum_{\nu \in \mathcal{N}_h} \langle v_h \rangle_\nu \varphi_\nu \,,$$

$$\langle v_h \rangle_\nu \coloneqq \begin{cases} \frac{1}{\mathrm{card}(\mathcal{T}_h(\nu))} \sum_{T \in \mathcal{T}_h(\nu)} (v_h|_T)(\nu) & \text{if } \nu \in \Omega \cup \Gamma_N \,, \\ 0 & \text{if } \nu \in \Gamma_D \,, \end{cases}$$

where $\mathcal{T}_h(\nu) \coloneqq \{T \in \mathcal{T}_h \mid \nu \in T\}$ and $(\varphi_\nu)_{\nu \in \mathcal{N}_h}$ nodal basis of $\mathcal{S}^1(\mathcal{T}_h)$.



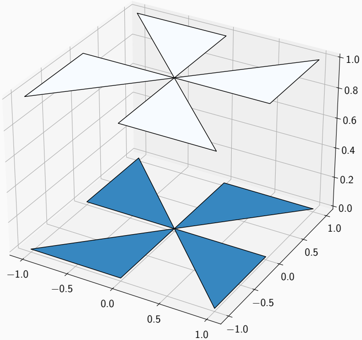


Figure: $v_h \in \mathbb{P}^0(\mathcal{T}_h)$.

- **Node-averaging operator:** $\Pi_h^{av}\colon \mathbb{P}^1(\mathcal{T}_h) \to \mathcal{S}^1_D(\mathcal{T}_h)$, for every $v_h \in \mathbb{P}^1(\mathcal{T}_h)$ defined by

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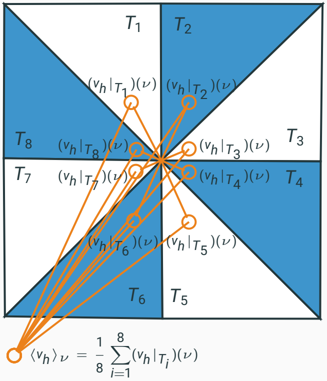

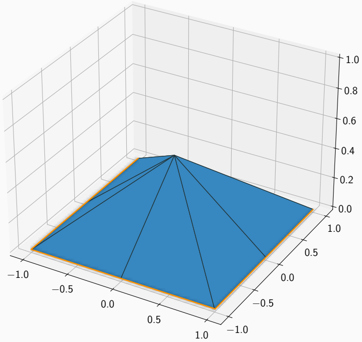

Figure: $\Pi_h^{av} v_h \in \mathcal{S}^1_D(\mathcal{T}_h)$ if $\Gamma_D = \partial\Omega$.

- **Node-averaging operator:** $\Pi_h^{av} : \mathbb{P}^1(\mathcal{T}_h) \to \mathcal{S}^1_D(\mathcal{T}_h)$, for every $v_h \in \mathbb{P}^1(\mathcal{T}_h)$ defined by

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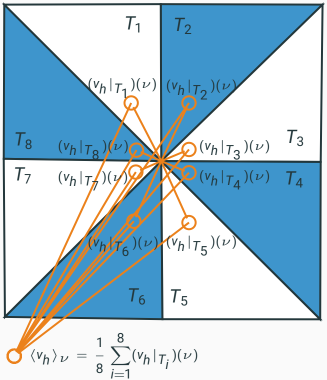

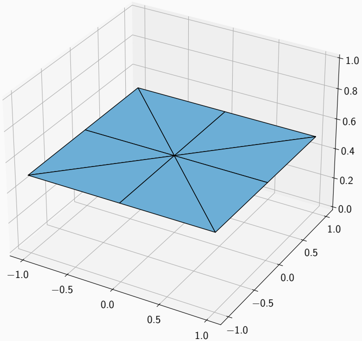

Figure: $\Pi_h^{av} v_h \in \mathcal{S}^1_D(\mathcal{T}_h)$ if $\Gamma_D = \emptyset$.

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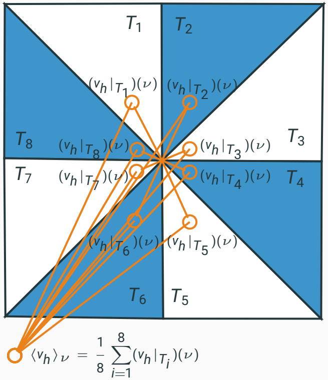


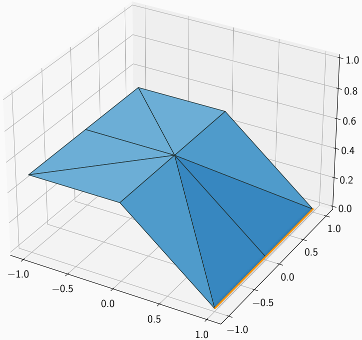


Figure: $\Pi_h^{av} v_h \in \mathcal{S}^1_D(\mathcal{T}_h)$ if $\Gamma_D = \{1\} \times [-1, 1]$.

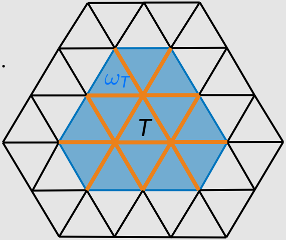
Lemma: ((local) best-approximation property)

For every $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$, it holds that

$$\int_T |\nabla_h v_h - \nabla \Pi_h^{av} v_h|^2 \,\mathrm{d}x \lesssim \inf_{v \in W^{1,2}_D(\Omega)} \int_{\omega_T} |\nabla_h v_h - \nabla v|^2 \,\mathrm{d}x\,.$$

In particular, for every $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ and $v \in W^{1,2}_D(\Omega)$, it holds that

$$\int_T |\nabla \Pi_h^{av} v_h - \nabla v|^2 \,\mathrm{d}x \lesssim \int_{\omega_T} |\nabla_h v_h - \nabla v|^2 \,\mathrm{d}x\,.$$



- **Proof.** (*cf.* [1, Bartels, K.]) For every $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ and $v \in W^{1,2}_D(\Omega)$, it holds that

$$\begin{aligned}\int_T |\nabla_h v_h - \nabla \Pi_h^{av} v_h|^2 \,\mathrm{d}x &\lesssim \sum_{S \in \mathcal{S}_h^i \cup \mathcal{S}_h^{\Gamma_D} \,:\, S \cap T \neq \emptyset} \int_S h_T^{-1} |[\![v_h]\!]_S|^2 \,\mathrm{d}s \\ &= \sum_{S \in \mathcal{S}_h^i \cup \mathcal{S}_h^{\Gamma_D} \,:\, S \cap T \neq \emptyset} \int_S h_T^{-1} \left| [\![v_h - v]\!]_S - \int_S [\![v_h - v]\!]_S \,\mathrm{d}s \right|^2 \mathrm{d}s \\ &\lesssim \int_{\omega_T} |\nabla_h v_h - \nabla v|^2 \,\mathrm{d}x\,. \qquad \blacksquare\end{aligned}$$

Adaptive algorithm

Algorithm: (SOLVE–ESTIMATE–MARK–REFINE)

Let $\varepsilon_{\text{STOP}} > 0$, $\theta \in (0,1]$, and $\mathcal{T}_0$ an initial triangulation. Then, for every $k \geq 0$:

('SOLVE') Compute discrete primal solution and discrete dual solution

$$u_k^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k) \qquad \& \qquad z_k^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_k)\,;$$

('ESTIMATE') Post-process $u_k^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k)$ and $z_k^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_k)$ to obtain

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for all $T \in \mathcal{T}_k$. If $\eta^2_{\text{gap}}(\overline{u}_k^{cr},\overline{z}_k^{rt}) \leq \varepsilon_{\text{STOP}}$, then STOP;

('MARK') Use *marking strategy* to find $\mathcal{M}_k \subseteq \mathcal{T}_k$ on which error is *'concentrated'*;

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Increase $k \to k+1$ and continue with ('SOLVE').

- **Marking strategies:** For *bulk parameter* $\theta \in (0,1]$, find (minimal wrt. cardinality)

$$\mathcal{M}_k \subseteq \mathcal{T}_k\,,$$

s.t. *bulk criterion* is satisfied.

- **Max marking:** Choose $\mathcal{M}_k \subseteq \mathcal{T}_k$ s.t.

$$\min_{T\in\mathcal{M}_k} \eta^2_{\mathrm{gap},T}(\overline{u}_k^{cr},\overline{z}_k^{rt}) \geq \theta^2 \max_{T\in\mathcal{T}_h} \eta^2_{\mathrm{gap},T}(\overline{u}_k^{cr},\overline{z}_k^{rt})\,.$$

- **Dörfler marking:** (*cf.* [3, Dörfler, '96]) Choose $\mathcal{M}_k \subseteq \mathcal{T}_k$ s.t.

$$\sum_{T\in\mathcal{M}_k} \eta^2_{\mathrm{gap},T}(\overline{u}_k^{cr},\overline{z}_k^{rt}) \geq \theta \sum_{T\in\mathcal{T}_k} \eta^2_{\mathrm{gap},T}(\overline{u}_k^{cr},\overline{z}_k^{rt})\,.$$

Construction: For sorted set of triangles $\mathcal{T}_k^{\mathrm{sorted}} := \{T_1,\dots,T_{\mathrm{card}(\mathcal{T}_k)}\}$, where

$$i \leq j \qquad :\Leftrightarrow \qquad \eta^2_{\mathrm{gap},T_i}(\overline{u}_k^{cr},\overline{z}_k^{rt}) \geq \eta^2_{\mathrm{gap},T_j}(\overline{u}_k^{cr},\overline{z}_k^{rt})\,,$$

we define $\mathcal{M}_k \coloneqq \{T_i \in \mathcal{T}_k^{\mathrm{sorted}} \mid i = 1,\dots,N_k^*\}$, where

$$N_k^* \coloneqq \min\left\{ i = 1,\dots,\mathrm{card}(\mathcal{T}_h) \;\middle|\; \sum_{i=1}^{N_k^*} \eta^2_{\mathrm{gap},T_i}(\overline{u}_k^{cr},\overline{z}_k^{rt}) \geq \theta \sum_{T\in\mathcal{T}_k} \eta^2_{\mathrm{gap},T}(\overline{u}_k^{cr},\overline{z}_k^{rt}) \right\}.$$

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for all $T \in \mathcal{T}_k$. If $\eta^2_{\mathrm{gap}}(\overline{u}_k^{cr},\overline{z}_k^{rt}) \leq \varepsilon_{\mathrm{STOP}}$, then STOP;

('MARK') Use *marking strategy* to find $\mathcal{M}_k \subseteq \mathcal{T}_k$ on which error is *'concentrated'*;

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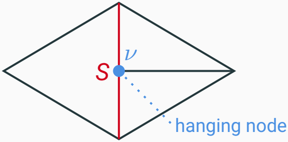
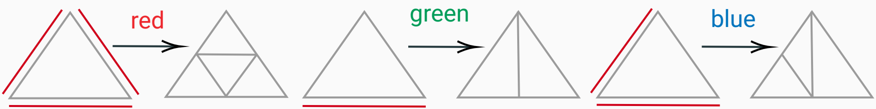
Increase $k \to k+1$ and continue with ('SOLVE').

REFINE

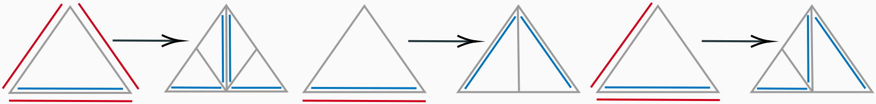
- **Regular/irregular refinement:**
  - **Regular refinement:** Refine all marked elements (*i.e.*, all $T \in \mathcal{M}_k$);
  - **Irregular refinement:** Refine further unmarked elements to avoid *hanging nodes*, *i.e.*,



- **Refinement routines:**
  - **Red-Green-Blue refinement:** (*cf.* [2, Carstensen, '04])



  - **Newest-Vertex-Bisection:** (*cf.* [4, Mitchell, '88])



Examples

Algorithm: (SOLVE–ESTIMATE–MARK–REFINE)

Let $\varepsilon_{\mathrm{STOP}} > 0$, $\theta \in (0,1]$, and $\mathcal{T}_0$ an initial triangulation. Then, for every $k \geq 0$:

('SOLVE') Compute $u_k^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k)$ using *linear solver* and $z_k^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_k)$ using *discrete reconstruction formula*.

('ESTIMATE') Post-process $u_k^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k)$ via *node-averaging* to obtain

$$\overline{u}_k^{cr} \coloneqq \Pi_{h_k}^{av} u_k^{cr} \in \mathrm{dom}(I)\,;$$

Compute *(local) refinement indicators* $\{\eta^2_{\mathrm{gap},T}(\overline{u}_k^{cr}, z_k^{rt})\}_{T\in\mathcal{T}_k}$, defined by

$$\eta^2_{\mathrm{gap},T}(\overline{u}_k^{cr}, z_k^{rt}) \coloneqq \frac{1}{2}\int_T |z_k^{rt} - \nabla \overline{u}_k^{cr}|^2\,\mathrm{d}x$$

for all $T \in \mathcal{T}_k$. If $\eta^2_{\mathrm{gap}}(\overline{u}_k^{cr}, z_k^{rt}) \leq \varepsilon_{\mathrm{STOP}}$, then STOP;

('MARK') Use *Dörfler marking* to find $\mathcal{M}_k \subseteq \mathcal{T}_k$ on which error is *'concentrated'*;

('REFINE') Perform red-green-blue-refinement of $\mathcal{T}_k$ to get $\mathcal{T}_{k+1}$ s.t. each $T \in \mathcal{M}_k$ is red-refined in $\mathcal{T}_{k+1}$.

Increase $k \to k+1$ and continue with ('SOLVE').

- **Implementation details wrt. step ('SOLVE'):**
  - **Computation of discrete primal solution:** Compute

$$u_k^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k)\,,$$

s.t. for every $v_k \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k)$, it holds that

$$\int_\Omega \nabla_{h_k} u_k^{cr} \cdot \nabla_{h_k} v_k \,\mathrm{d}x = \int_\Omega f_{h_k}\, \Pi_{h_k} v_k \,\mathrm{d}x\,,$$

using, *e.g.*, *CG method preconditioned with incomplete LU factorization*.

Computational costs:

$$N_k \coloneqq \mathtt{ndof}(\mathcal{S}_D^{1,cr}(\mathcal{T}_k)) = \mathrm{card}(\mathcal{S}_h \setminus \mathcal{S}_h^{\Gamma_D})\,.$$

  - **Computation of discrete dual solution:** Compute

$$z_k^{rt} \coloneqq \nabla_{h_k} u_k^{cr} - \frac{f_{h_k}}{d}(\mathrm{id}_{\mathbb{R}^d} - \Pi_{h_k}\mathrm{id}_{\mathbb{R}^d}) \in \mathcal{R}T_N^0(\mathcal{T}_k)\,,$$

i.e., $z_k^{rt} \in \mathrm{dom}(-D) \cap W_N^2(\mathrm{div};\Omega)$ if $f = f_{h_k} \in \mathbb{P}^0(\mathcal{T}_k)$.

Computational costs: for free!

Example with corner singularity:

Let $\Omega := (-1,1)^2 \setminus ([0,1]\times[-1,0])$, $\Gamma_D = \partial\Omega$, and $f = 1 \in L^2(\Omega)$. Then, $u \in W^{1,2}_D(\Omega)$ with

$$z = \nabla u \in (W^{\frac{2}{3},2}(\Omega))^2\,.$$

⇒ **Expected for uniform mesh-refinement (*i.e.*, $\theta = 1$):** $\eta^2_{\mathrm{gap}}(\overline{u}^{cr}_k, z^{rt}_k) \sim h_k^{\frac{4}{3}} \sim N_k^{-\frac{2}{3}}$.

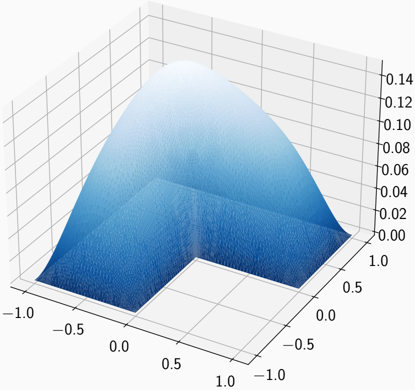


Figure: $u \in W^{1,2}_D(\Omega)$.

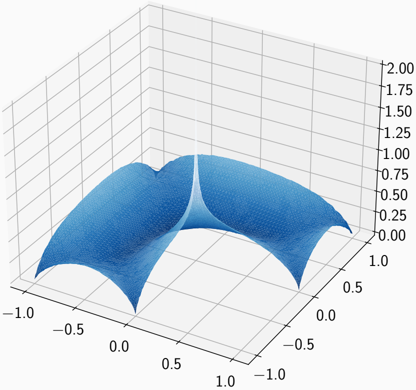
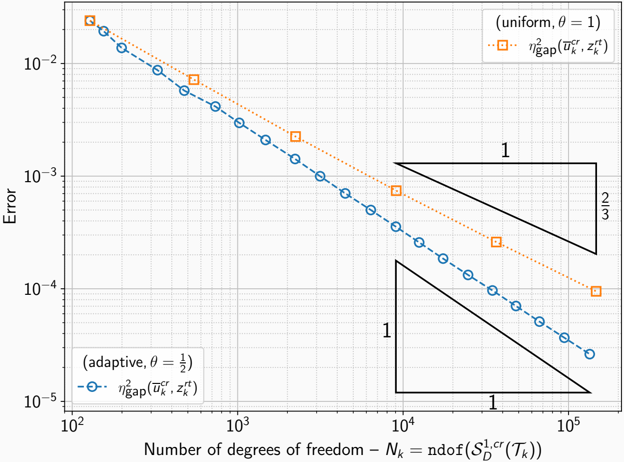
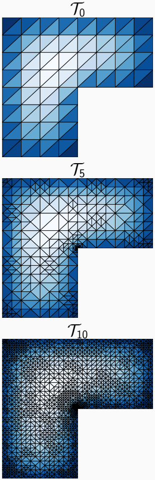


Figure: $|\nabla u| \in L^2(\Omega)$.

Examples: Poisson problem





- **Observations:**
  - → Adaptive mesh refinement yields optimal convergence rate $h_k^2 \sim N_k^{-1}$.
  - → Uniform mesh refinement yields reduced convergence rate $h_k^{\frac{4}{3}} \sim N_k^{-\frac{2}{3}}$.

Examples: p-Dirichlet problem

Algorithm: (SOLVE–ESTIMATE–MARK–REFINE)

Let $\varepsilon_{\mathrm{STOP}} > 0$, $\theta \in (0,1]$, and $\mathcal{T}_0$ an initial triangulation. Then, for every $k \geq 0$:

('SOLVE') Compute $u_k^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k)$ using *non-linear solver* and $z_k^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_k)$ using *discrete reconstruction formula*.

('ESTIMATE') Post-process $u_k^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k)$ via *node-averaging* to obtain

$$\overline{u}_k^{cr} \coloneqq \Pi_{h_k}^{av} u_k^{cr} \in \mathrm{dom}(I)\,;$$

Compute *(local) refinement indicators* $\{\eta^2_{\mathrm{gap},T}(\overline{u}_k^{cr}, z_k^{rt})\}_{T\in\mathcal{T}_k}$, defined by

$$\eta^2_{\mathrm{gap},T}(\overline{u}_k^{cr}, z_k^{rt}) \coloneqq \int_T \left\{\frac{1}{p'}|z_k^{rt}|^{p'} - z_k^{rt}\cdot\nabla\overline{u}_k^{cr} + \frac{1}{p}|\nabla\overline{u}_k^{cr}|^p\right\}\,\mathrm{d}x$$

for all $T \in \mathcal{T}_k$. If $\eta^2_{\mathrm{gap}}(\overline{u}_k^{cr}, z_k^{rt}) \leq \varepsilon_{\mathrm{STOP}}$, then STOP;

('MARK') Use *Dörfler marking* to find $\mathcal{M}_k \subseteq \mathcal{T}_k$ on which error is *'concentrated'*;

('REFINE') Perform red-green-blue-refinement of $\mathcal{T}_k$ to get $\mathcal{T}_{k+1}$ s.t. each $T \in \mathcal{M}_k$ is red-refined in $\mathcal{T}_{k+1}$.

Increase $k \to k+1$ and continue with ('SOLVE').

Examples: p-Dirichlet problem

- **Implementation details wrt. step ('SOLVE'):**
  - **Computation of discrete primal solution:** Compute

$$u_k^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k)\,,$$

s.t. for every $v_k \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k)$, it holds that

$$\int_\Omega D\phi_k(\nabla_{h_k} u_k^{cr}) \cdot \nabla_{h_k} v_k \,\mathrm{d}x = \int_\Omega f_{h_k}\, \Pi_{h_k} v_k \,\mathrm{d}x\,,$$

where $\phi_k \in C^1(\mathbb{R}^d)$ is defined by

$$\phi_k(0) = 0 \qquad \text{and} \qquad D\phi_k(t) = (h_k^2 + |t|)^{p-2} t \quad \text{for all } t \in \mathbb{R}^d\,,$$

using, *e.g.*, *Newton scheme with line search*.

Computational costs: (in each Newton step)

$$N_k \coloneqq \mathtt{ndof}(\mathcal{S}_D^{1,cr}(\mathcal{T}_k)) = \mathrm{card}(\mathcal{S}_h \setminus \mathcal{S}_h^{\Gamma_D})\,.$$

  - **Computation of discrete dual solution:** Compute

$$z_k^{rt} \coloneqq D\phi_k(\nabla_{h_k} u_k^{cr}) - \frac{f_{h_k}}{d}(\mathrm{id}_{\mathbb{R}^d} - \Pi_{h_k}\mathrm{id}_{\mathbb{R}^d}) \in \mathcal{R}T_N^0(\mathcal{T}_k)\,,$$

i.e., $z_k^{rt} \in \mathrm{dom}(-D) \cap W_N^{p'}(\mathrm{div};\Omega)$ if $f = f_{h_k} \in \mathbb{P}^0(\mathcal{T}_k)$.

Computational costs: for free!

- **Example with corner singularity:**

  Let $\Omega = (-1,1)^2 \setminus ([0,1]\times[-1,0])$, $\Gamma_D = \partial\Omega$, and $f = 1 \in L^{p'}(\Omega)$, $p \in (1,\infty)$. Then, $u \in W^{1,p}_D(\Omega)$ with

$$V(\nabla u) = V^*(z) \in (W^{\frac{2}{3},2}(\Omega))^2 \,.$$

⇒ **Expected for uniform mesh-refinement (*i.e.*, $\theta = 1$):** $\eta^2_{\mathrm{gap}}(\overline{u}^{cr}_k, z^{rt}_k) \sim h_k^{\frac{4}{3}} \sim N_k^{-\frac{2}{3}}$.

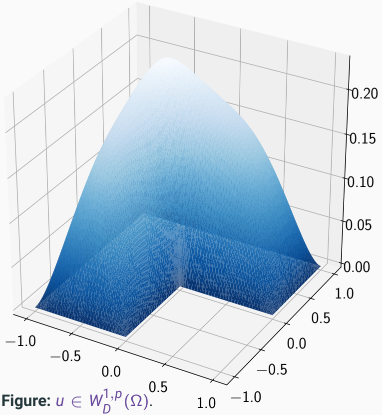


Figure: $u \in W^{1,p}_D(\Omega)$.

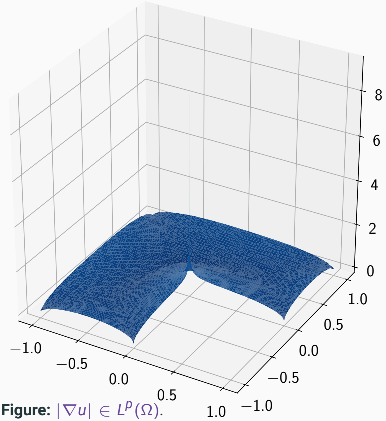
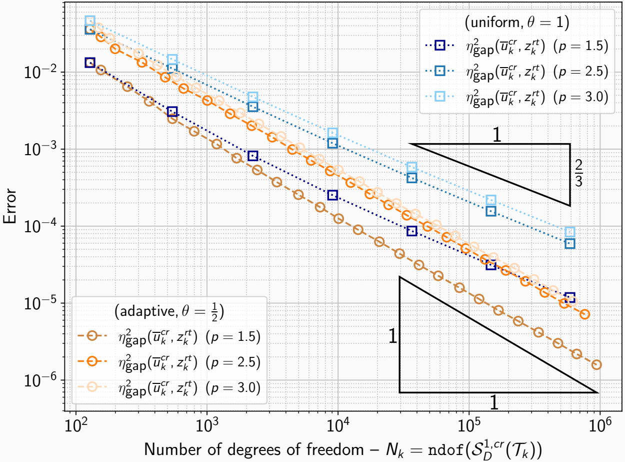
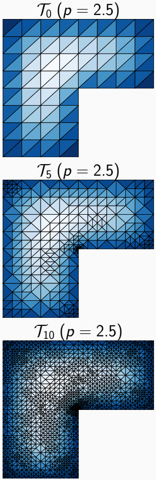


Figure: $|\nabla u| \in L^p(\Omega)$.

Examples: p-Dirichlet problem





- **Observations:**
  - → Adaptive mesh refinement yields optimal convergence rate $h_k^2 \sim N_k^{-1}$.
  - → Uniform mesh refinement yields reduced convergence rate $h_k \sim N_k^{-\frac{2}{3}}$.

Examples: Obstacle problem

Algorithm: (SOLVE–ESTIMATE–MARK–REFINE)

Let $\varepsilon_{\text{STOP}} > 0$, $\theta \in (0,1]$, and $\mathcal{T}_0$ an initial triangulation. Then, for every $k \geq 0$:

('SOLVE') Compute $u_k^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k)$ using *non-linear solver* and $z_k^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_k)$ using *discrete reconstruction formula*.

('ESTIMATE') Post-process $u_k^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k)$ via *node-averaging* and *cut-off* to obtain

$$\overline{u}_k^{cr} := \max\{0, \Pi_{h_k}^{av} u_k^{cr}\} \in \operatorname{dom}(I)\,;$$

Compute *(local) refinement indicators* $\{\eta^2_{\text{gap},T}(\overline{u}_k^{cr}, z_k^{rt})\}_{T\in\mathcal{T}_k}$, defined by

$$\eta^2_{\text{gap},T}(\overline{u}_k^{cr}, z_k^{rt}) := \left\{\begin{aligned} &\frac{1}{2}\int_T |z_k^{rt} - \nabla \overline{u}_k^{cr}|^2\,\mathrm{d}x \\ &-\int_T (f_{h_k} + \operatorname{div} z_k^{rt})\,\overline{u}_k^{cr}\,\mathrm{d}x \end{aligned}\right\}$$

for all $T \in \mathcal{T}_k$. If $\eta^2_{\text{gap}}(\overline{u}_k^{cr}, z_k^{rt}) \leq \varepsilon_{\text{STOP}}$, then STOP;

('MARK') Use *Dörfler marking* to find $\mathcal{M}_k \subseteq \mathcal{T}_k$ on which error is *'concentrated'*;

('REFINE') Perform red-green-blue-refinement of $\mathcal{T}_k$ to get $\mathcal{T}_{k+1}$ s.t. each $T \in \mathcal{M}_k$ is red-refined in $\mathcal{T}_{k+1}$.

Increase $k \to k+1$ and continue with ('SOLVE').

Examples: Obstacle problem

- **Implementation details wrt. ('SOLVE'):**
  - **Computation of discrete primal solution:** Compute

$$u_k^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k) \qquad \& \qquad \lambda_k^{cr} \in \Pi_{h_k}(\mathcal{S}_D^{1,cr}(\mathcal{T}_k))$$

s.t. $\lambda_k^{cr} \leq 0$ a.e. in Ω and for every $v_k \in \mathcal{S}_D^{1,cr}(\mathcal{T}_k)$, it holds that

$$\int_\Omega \nabla_{h_k} u_k^{cr} \cdot \nabla_{h_k} v_k \,\mathrm{d}x = \int_\Omega (f_{h_k} - \lambda_k^{cr})\, \Pi_{h_k} v_k \,\mathrm{d}x\,,$$

using, *e.g.*, *primal-dual active set strategy*, *i.e.*, *semi-smooth Newton method*.

Computational costs: (in each semi-smooth Newton step)

$$N_k = \mathtt{ndof}(\mathcal{S}_D^{1,cr}(\mathcal{T}_k)) + \mathtt{ndof}(\Pi_{h_k}(\mathcal{S}_D^{1,cr}(\mathcal{T}_k)))\,.$$

  - **Computation of discrete dual solution:** Compute

$$z_k^{rt} \coloneqq \nabla_{h_k} u_k^{cr} - \frac{f_{h_k} - \lambda_k^{cr}}{d}(\mathrm{id}_{\mathbb{R}^d} - \Pi_{h_k}\mathrm{id}_{\mathbb{R}^d}) \in \mathcal{RT}_N^0(\mathcal{T}_k)\,.$$

i.e., $z_k^{rt} \in \mathrm{dom}(-D) \cap W_N^2(\mathrm{div};\Omega)$ if $f = f_{h_k} \in \mathbb{P}^0(\mathcal{T}_k)$.

Computational costs: for free!

- **Example with corner singularity:**

  Let $\Omega := (-1,1)^2$, $\Gamma_D = \partial\Omega$, $f = 1 \in L^2(\Omega)$, and $\chi = 1 - |\cdot|_\infty \in \mathcal{S}^1(\mathcal{T}_0)$. Then, $u \in W^{1,2}_D(\Omega)$ with

$$z = \nabla u \in (W^{\frac{1}{2},2}(\Omega))^2 \,.$$

$\Rightarrow$ **Expected for uniform mesh-refinement (*i.e.*, $\theta = 1$):** $\eta^2_{\mathrm{gap}}(\overline{u}^{cr}_k, z^{rt}_k) \sim h_k \sim N_k^{-\frac{1}{2}}$.

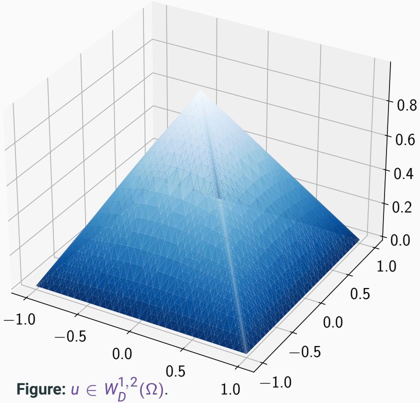


Figure: $u \in W^{1,2}_D(\Omega)$.

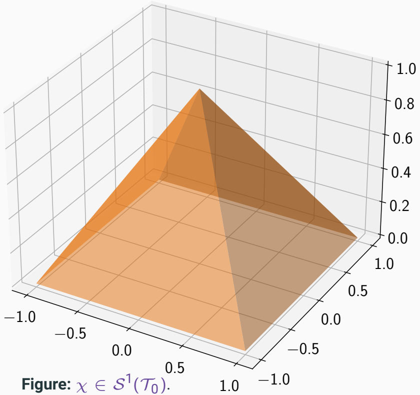
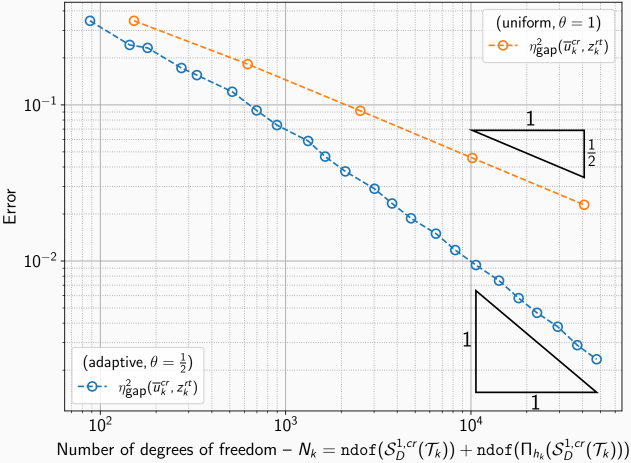
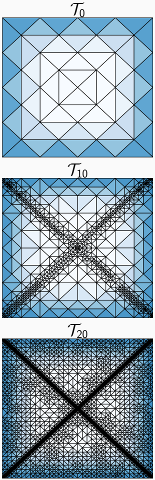


Figure: $\chi \in \mathcal{S}^1(\mathcal{T}_0)$.





Observations:

→ Adaptive mesh refinement yields optimal convergence rate $h_k^2 \sim N_k^{-1}$.

→ Uniform mesh refinement yields reduced convergence rate $h_k \sim N_k^{\frac{1}{2}}$.

Conclusions

- ***A posteriori* error analysis:** (on the basis of convex duality)
  - *Primal-dual gap identity*: for every $v \in \mathrm{dom}(I)$ and $y \in \mathrm{dom}(-D)$, it holds that

$$\rho^2_{\mathrm{tot}}(v,y) = \eta^2_{\mathrm{gap}}(v,y)\,.$$

  - *Advantages/Challenges*:

$$\left.\begin{array}{ll} \checkmark\ \text{systematic;} & \checkmark\ \text{widely applicable;} \\ \checkmark\ \text{simple;} & \checkmark\ \text{identical;} \end{array}\right\} \quad \text{vs.} \quad \textbf{?}\ \text{numerically practicable.}$$

- ***A posteriori* error analysis:** (on the basis of discrete convex duality)
  - *Discrete reconstruction formulas*: imply numerical practicability.

$$\begin{aligned} z_h^{rt} &= D_t\phi_h(\cdot,\nabla_h u_h^{cr}) + \frac{D_t\psi_h(\cdot,\Pi_h u_h^{cr})}{d}(\mathrm{id}_{\mathbb{R}^d} - \Pi_h \mathrm{id}_{\mathbb{R}^d})\,,\\ u_h^{cr} &= D_t\psi_h^*(\cdot,\operatorname{div} z_h^{rt}) + D_t\phi_h^*(\cdot,\Pi_h z_h^{rt})\cdot(\mathrm{id}_{\mathbb{R}^d} - \Pi_h \mathrm{id}_{\mathbb{R}^d})\,. \end{aligned}$$

A priori error identity on the basis of discrete convex duality

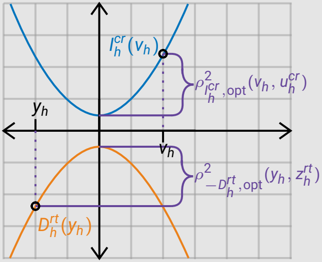
Discrete primal-dual gap identity

Definition: (discrete primal-dual total error & discrete primal-dual gap estimator)

- The *discrete primal-dual total error* $\rho^2_{\mathrm{tot},h}\colon \mathrm{dom}(I_h^{cr}) \times \mathrm{dom}(-D_h^{rt}) \to [0,+\infty)$ is defined by

$$\rho^2_{\mathrm{tot},h}(v_h, y_h) \coloneqq \rho^2_{I_h^{cr},\mathrm{opt}}(v_h, u_h^{cr}) + \rho^2_{-D_h^{rt},\mathrm{opt}}(y_h, z_h^{rt})$$

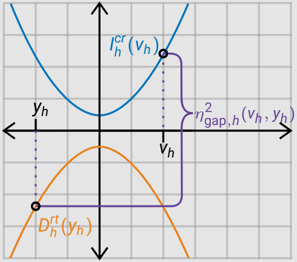
for all $(v_h, y_h)^\top \in \mathrm{dom}(I_h^{cr}) \times \mathrm{dom}(-D_h^{rt})$.



- The *discrete primal-dual gap estimator* $\eta^2_{\mathrm{gap},h}\colon \mathrm{dom}(I_h^{cr}) \times \mathrm{dom}(-D_h^{rt}) \to [0,+\infty)$ is defined by

$$\eta^2_{\mathrm{gap},h}(v_h, y_h) \coloneqq I_h^{cr}(v_h) - D_h^{rt}(y_h)$$

for all $(v_h, y_h)^\top \in \mathrm{dom}(I_h^{cr}) \times \mathrm{dom}(-D_h^{rt})$.



Discrete primal-dual gap identity

Theorem: (discrete primal-dual gap identity)

If a *discrete strong duality relation* applies, *i.e.*,

$$I_h^{cr}(u_h^{cr}) = D_h^{rt}(z_h^{rt})\,,$$

then for every $(v_h, y_h)^\top \in \operatorname{dom}(I_h^{cr}) \times \operatorname{dom}(-D_h^{rt})$, it holds that

$$\rho^2_{\mathrm{tot},h}(v_h, y_h) = \eta^2_{\mathrm{gap},h}(v_h, y_h)\,.$$

Proof. For every $(v_h, y_h)^\top \in \operatorname{dom}(I_h^{cr}) \times \operatorname{dom}(-D_h^{rt})$, we have that

$$\begin{aligned}
\rho^2_{\mathrm{tot},h}(v_h, y_h) &:= \rho^2_{I_h^{cr},\mathrm{opt}}(v_h, u_h^{cr}) + \rho^2_{-D_h^{rt},\mathrm{opt}}(y_h, z_h^{rt}) \\
&:= [I_h^{cr}(v_h) - I_h^{cr}(u_h^{cr})] \\
&\quad + [D_h^{rt}(z_h^{rt}) - D_h^{rt}(y_h)] \\
&= I_h^{cr}(v_h) - D_h^{rt}(y_h) \\
&\quad - \underbrace{[D_h^{rt}(z_h^{rt}) - I_h^{cr}(u_h^{cr})]}_{=0} \\
&= \eta^2_{\mathrm{gap},h}(v_h, y_h)\,. \qquad \blacksquare
\end{aligned}$$

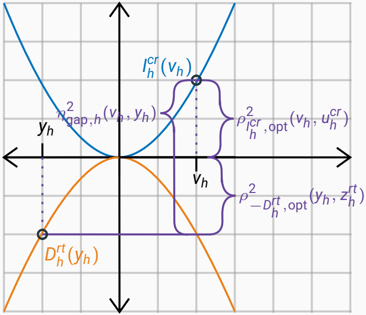


Figure: Discrete primal-dual gap identity in 1D.

Examples: Poisson problem

Theorem: (*a priori* error identity)

If $u \in W_D^{1+s,2}(\Omega)$, $s \in [0,1]$, *i.e.*, $z \in (W^{s,2}(\Omega))^d$, and $f_h = \Pi_h f \in \mathbb{P}^0(\mathcal{T}_h)$, then

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla_h \Pi_h^{cr} u - \nabla_h u_h^{cr}|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\Pi_h \Pi_h^{rt} z - \Pi_h z_h^{rt}|^2\,\mathrm{d}x \\ = \frac{1}{2}\int_\Omega |\Pi_h z - \Pi_h \Pi_h^{rt} z|^2\,\mathrm{d}x \lesssim h^{2s}\,[z]_{s,2}^2\,.\end{aligned}$$

Proof.

- For every $v_h \in \mathrm{dom}(I_h^{cr})$ and $y_h \in \mathrm{dom}(-D_h^{rt})$, we have that

$$\rho^2_{I_h^{cr},\mathrm{opt}}(v_h, u_h^{cr}) = \frac{1}{2}\int_\Omega |\nabla_h v_h - \nabla_h u_h^{cr}|^2\,\mathrm{d}x\,,$$

$$\rho^2_{-D_h^{rt},\mathrm{opt}}(y_h, z_h^{rt}) = \frac{1}{2}\int_\Omega |\Pi_h y_h - \Pi_h z_h^{rt}|^2\,\mathrm{d}x\,.$$

- For every $v_h \in \mathrm{dom}(I_h^{cr})$ and $y_h \in \mathrm{dom}(-D_h^{rt})$, we have that

$$\eta^2_{\mathrm{gap},h}(v_h, y_h) = \frac{1}{2}\int_\Omega |\nabla_h v_h - \Pi_h y_h|^2\,\mathrm{d}x\,.$$

Examples: Poisson problem

- It holds that $\Pi_h^{rt} z \in \mathrm{dom}(-D_h^{rt})$, since

$$\begin{aligned} \operatorname{div}\Pi_h^{rt} z &= \Pi_h \operatorname{div} z = \Pi_h[-f] = -f_h && \text{a.e. in } \Omega\,,\\ \Pi_h^{rt} z\cdot n &= \pi_h[z\cdot n] = 0 && \text{a.e. in } \Gamma_N\,. \end{aligned}$$

- It holds that $\Pi_h^{cr} u \in \mathrm{dom}(I_h^{cr})$, since

$$\begin{aligned} \nabla_h \Pi_h^{cr} u &= \Pi_h \nabla u = \Pi_h z && \text{a.e. in } \Omega\,,\\ \pi_h \Pi_h^{cr} u &= \pi_h u = 0 && \text{a.e. in } \Gamma_D\,. \end{aligned}$$

→ For every $v_h = \Pi_h^{cr} u \in \mathrm{dom}(I_h^{cr})$ and $y_h = \Pi_h^{rt} z \in \mathrm{dom}(-D_h^{rt})$, we conclude that

$$\begin{aligned} \frac{1}{2}\int_\Omega |\nabla_h \Pi_h^{cr} u - \nabla_h u_h^{cr}|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\Pi_h \Pi_h^{rt} z - \Pi_h z_h^{rt}|^2\,\mathrm{d}x &= \frac{1}{2}\int_\Omega |\nabla_h \Pi_h^{cr} u - \Pi_h \Pi_h^{rt} z|^2\,\mathrm{d}x\\ &= \frac{1}{2}\int_\Omega |\Pi_h z - \Pi_h \Pi_h^{rt} z|^2\,\mathrm{d}x\\ &\leq \frac{1}{2}\int_\Omega |z - \Pi_h^{rt} z|^2\,\mathrm{d}x\\ &\lesssim h^{2s}\,[z]_{s,2}^2\,. \end{aligned}$$

■

- **Example with central singularity:**

  Let $\Omega := (-1,1)^2$, $\Gamma_D = \partial\Omega$, and $f \in L^2(\Omega)$ s.t. $u \in W_D^{1,2}(\Omega)$ is given via

  $$u(x_1,x_2) := (1-x_1^2)(1-x_2^2)|(x_1,x_2)^\top|^{s+\delta},$$

  where $s \in (0,1]$ and $\delta = 1.0 \times 10^{-5}$, so that $z = \nabla u \in (W^{s,2}(\Omega))^2$.

$\Rightarrow$ **Expected for uniform mesh-refinement (*i.e.*, $\theta = 1$):** $\eta^2_{\mathrm{gap},h_k}(\Pi_{h_k}^{cr} u, \Pi_{h_k}^{rt} z) \sim h_k^{2s} \sim N_k^{-s}$.

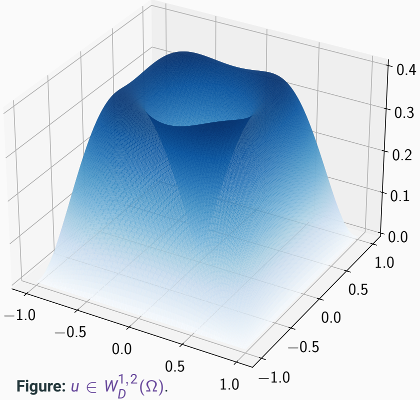


Figure: $u \in W_D^{1,2}(\Omega)$.

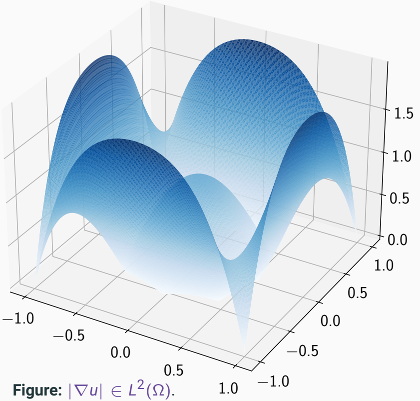
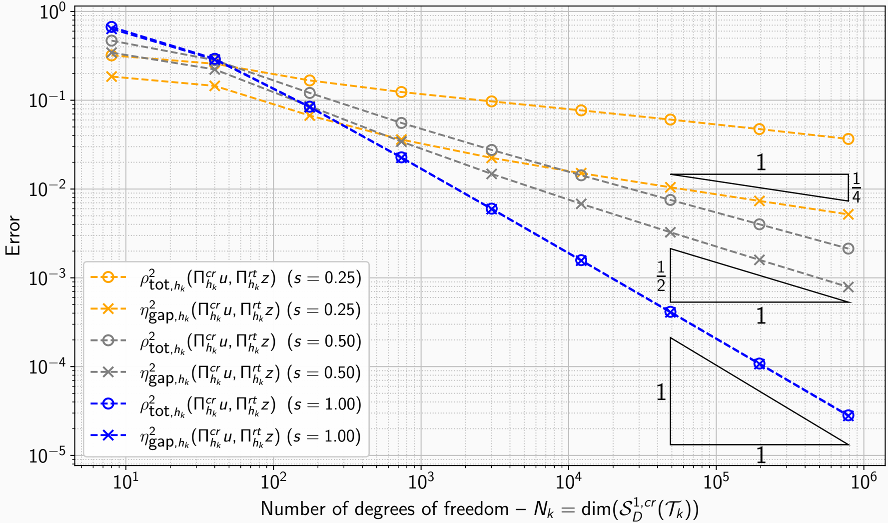


Figure: $|\nabla u| \in L^2(\Omega)$.

- **Experiment:**
  - **Results:**



Examples: p-Dirichlet problem

Theorem: (*a priori* error equivalence)

If $V(\nabla u) \in (W^{s,2}(\Omega))^d$, $s \in (0,1]$, *i.e.*, $V^*(z) \in (W^{s,2}(\Omega))^d$, and $f_h = \Pi_h f \in \mathbb{P}^0(\mathcal{T}_h)$, then

$$\int_\Omega |V(\nabla_h \Pi_h^{cr} u) - V(\nabla_h u_h^{cr})|^2 \,\mathrm{d}x + \int_\Omega |V^*(\Pi_h \Pi_h^{rt} z) - V^*(\Pi_h z_h^{rt})|^2 \,\mathrm{d}x$$

$$\sim \int_\Omega \Big\{ \frac{1}{p} |\Pi_h \nabla u|^p - \Pi_h \nabla u \cdot \Pi_h \Pi_h^{rt} z + \frac{1}{p'} |\Pi_h \Pi_h^{rt} z|^{p'} \Big\} \,\mathrm{d}x \lesssim h^{2s} \, [V^*(z)]_{s,2}^2 \,.$$

Proof.

- For every $v_h \in \mathrm{dom}(I_h^{cr})$ and $y_h \in \mathrm{dom}(-D_h^{rt})$, we have that

$$\rho^2_{I_h^{cr},\mathrm{opt}}(v_h, u_h^{cr}) \sim \int_\Omega |V(\nabla_h v_h) - V(\nabla_h u_h^{cr})|^2 \,\mathrm{d}x \,,$$

$$\rho^2_{-D_h^{rt},\mathrm{opt}}(y_h, z_h^{rt}) \sim \int_\Omega |V^*(\Pi_h y_h) - V^*(\Pi_h z_h^{rt})|^2 \,\mathrm{d}x \,.$$

- For every $v_h \in \mathrm{dom}(I_h^{cr})$ and $y_h \in \mathrm{dom}(-D_h^{rt})$, we have that

$$\eta^2_{\mathrm{gap},h}(v_h, y_h) = \int_\Omega \Big\{ \frac{1}{p} |\nabla_h v_h|^p - \nabla_h v_h \cdot \Pi_h y_h + \frac{1}{p'} |\Pi_h y_h|^{p'} \Big\} \,\mathrm{d}x \,.$$

Examples: p-Dirichlet problem

→ For every $v_h = \Pi_h^{cr} u \in \mathrm{dom}(I_h^{cr})$ and $y_h = \Pi_h^{rt} z \in \mathrm{dom}(-D_h^{rt})$, we conclude that

$$
\begin{aligned}
&\int_\Omega |V(\nabla_h \Pi_h^{cr} \nabla u) - V(\nabla_h u_h^{cr})|^2 \,\mathrm{d}x + \int_\Omega |V^*(\Pi_h \Pi_h^{rt} z) - V^*(\Pi_h z_h^{rt})|^2 \,\mathrm{d}x \\
\sim &\int_\Omega \Big\{ \frac{1}{p} |\nabla_h \Pi_h^{cr} u|^p - \nabla_h \Pi_h^{cr} u \cdot \Pi_h \Pi_h^{rt} z + \frac{1}{p'} |\Pi_h \Pi_h^{rt} z|^{p'} \Big\} \,\mathrm{d}x \\
\leq &\int_\Omega \Big\{ \underbrace{\frac{1}{p} |\nabla u|^p}_{= \nabla u \cdot z - \frac{1}{p'} |z|^{p'}} - \nabla u \cdot \Pi_h \Pi_h^{rt} z + \frac{1}{p'} |\Pi_h \Pi_h^{rt} z|^{p'} \Big\} \,\mathrm{d}x \\
= &\int_\Omega \Big\{ \underbrace{-\frac{1}{p'} |z|^{p'}}_{\leq -\frac{1}{p'} |\Pi_h \Pi_h^{rt} z|^{p'} - |\Pi_h \Pi_h^{rt} z|^{p'-2} \Pi_h \Pi_h^{rt} z \cdot (z - \Pi_h \Pi_h^{rt} z)} + \overbrace{\nabla u}^{= |z|^{p'-2} z} \cdot (z - \Pi_h \Pi_h^{rt} z) + \frac{1}{p'} |\Pi_h \Pi_h^{rt} z|^{p'} \Big\} \,\mathrm{d}x \\
\leq &\int_\Omega (|z|^{p'-2} z - |\Pi_h \Pi_h^{rt} z|^{p'-2} \Pi_h \Pi_h^{rt} z) \cdot (z - \Pi_h \Pi_h^{rt} z) \,\mathrm{d}x \\
\sim &\int_\Omega |V^*(z) - V^*(\Pi_h \Pi_h^{rt} z)|^2 \,\mathrm{d}x \\
\lesssim\; & h^{2s} \, [V^*(z)]_{s,2}^2 \,. \qquad \blacksquare
\end{aligned}
$$

Examples: p-Dirichlet problem

- **Example with central singularity:**

Let $\Omega := (-1,1)^2$, $\Gamma_D = \partial\Omega$, and $f \in L^{p'}(\Omega)$, $p \in (1,+\infty)$, s.t. $u \in W^{1,p}_D(\Omega)$ is given via

$$u(x_1,x_2) := (1-x_1^2)(1-x_2^2)|(x_1,x_2)^\top|^{1+2\frac{s-1}{p}+\delta},$$

where $s \in (0,1]$ and $\delta = 1.0\times 10^{-5}$, so that $V^*(z) = V(\nabla u) \in (W^{s,2}(\Omega))^2$.

$\Rightarrow$ **Expected for uniform mesh-refinement (*i.e.*, $\theta = 1$):** $\eta^2_{\mathrm{gap},h_k}(\Pi^{cr}_{h_k}u, \Pi^{rt}_{h_k}z) \sim h_k^{2s} \sim N_k^{-s}$.

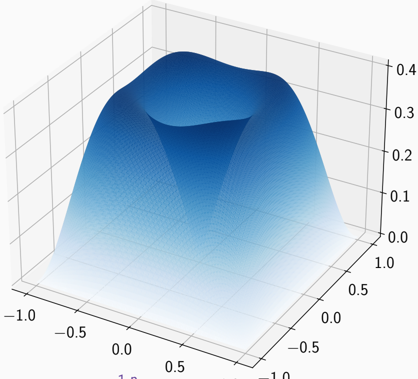


Figure: $u \in W^{1,p}_D(\Omega)$.

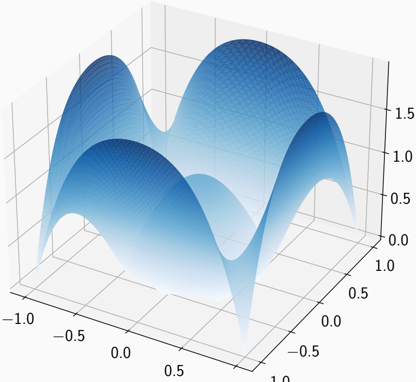
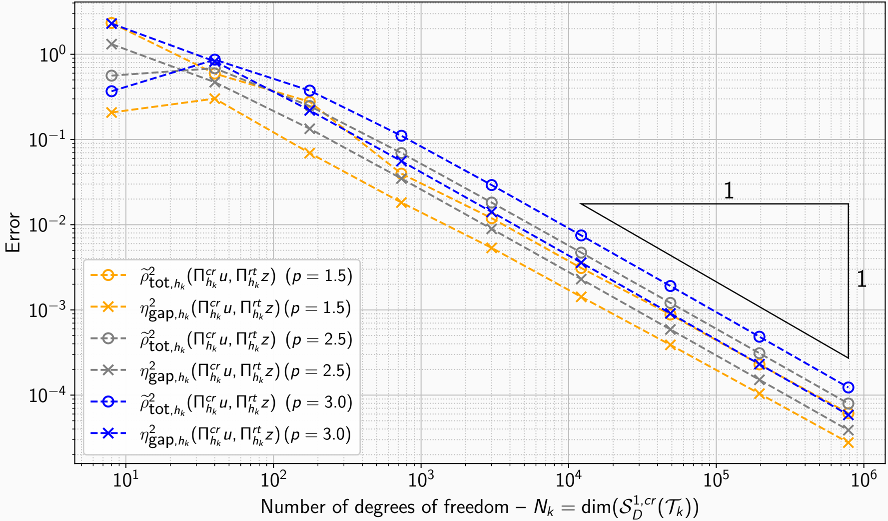


Figure: $|\nabla u| \in L^p(\Omega)$.

Examples: p-Dirichlet problem

- **Experiment:**
  - **Results:**



Examples: Obstacle problem

Theorem: (*a priori* error identity)

If $z \in (W^{s,2}(\Omega))^d \cap W^2_N(\mathrm{div};\Omega)$ and $f_h = \Pi_h f \in \mathbb{P}^0(\mathcal{T}_h)$, then

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla_h \Pi_h^{cr} u - \nabla_h u_h^{cr}|^2 \,\mathrm{d}x - \int_\Omega (f_h + \operatorname{div} z_h^{rt})\, \Pi_h \Pi_h^{cr} u \,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega |\Pi_h \Pi_h^{rt} z - \Pi_h z_h^{rt}|^2 \,\mathrm{d}x - \int_\Omega (f + \operatorname{div} z)\, \Pi_h u_h^{cr} \,\mathrm{d}x \\
&= \frac{1}{2}\int_\Omega |\Pi_h z - \Pi_h \Pi_h^{rt} z|^2 \,\mathrm{d}x + \int_\Omega (f + \operatorname{div} z)\,(u - \Pi_h \Pi_h^{cr} u) \,\mathrm{d}x \lesssim h^{2s}\,[z]^2_{s,2}\,.
\end{aligned}
$$

Proof.

- For every $v_h \in \mathrm{dom}(I_h^{cr})$ and $y_h \in \mathrm{dom}(-D_h^{rt})$, we have that

$$
\begin{aligned}
\rho^2_{I_h^{cr},\mathrm{opt}}(v_h, u_h^{cr}) &= \frac{1}{2}\int_\Omega |\nabla_h v_h - \nabla_h u_h^{cr}|^2 \,\mathrm{d}x - \int_\Omega (f_h + \operatorname{div} z_h^{rt})\, \Pi_h v_h \,\mathrm{d}x\,,\\
\rho^2_{-D_h^{rt},\mathrm{opt}}(y_h, z_h^{rt}) &= \frac{1}{2}\int_\Omega |\Pi_h y_h - \Pi_h z_h^{rt}|^2 \,\mathrm{d}x - \int_\Omega (f_h + \operatorname{div} y_h)\, \Pi_h u_h^{cr} \,\mathrm{d}x\,.
\end{aligned}
$$

- For every $v_h \in \mathrm{dom}(I_h^{cr})$ and $y_h \in \mathrm{dom}(-D_h^{rt})$, we have that

$$
\eta^2_{\mathrm{gap},h}(v_h, y_h) = \frac{1}{2}\int_\Omega |\nabla_h v_h - \Pi_h y_h|^2 \,\mathrm{d}x - \int_\Omega (f_h + \operatorname{div} y_h)\, \Pi_h v_h \,\mathrm{d}x\,.
$$

Examples: Obstacle problem

- It holds that $\Pi_h^{rt} z \in \mathrm{dom}(-D_h^{rt})$, since

$$\begin{aligned} f_h + \mathrm{div}\,\Pi_h^{rt} z &= \Pi_h(f + \mathrm{div}\,z) \leq 0 && \text{a.e. in } \Omega\,,\\ \Pi_h^{rt} z \cdot n &= \pi_h[z \cdot n] = 0 && \text{a.e. in } \Gamma_N\,. \end{aligned}$$

- It holds that $\Pi_h^{cr} u \in \mathrm{dom}(I_h^{cr})$, since

$$\begin{aligned} \nabla_h \Pi_h^{cr} u &= \Pi_h \nabla u = \Pi_h z && \text{a.e. in } \Omega\,,\\ \pi_h \Pi_h^{cr} u &= \pi_h u = 0 && \text{a.e. in } \Gamma_D\,,\\ \Pi_h \Pi_h^{cr} u &\geq 0 && \text{a.e. in } \Omega\,. \end{aligned}$$

$\Rightarrow$ For $v_h = \Pi_h^{cr} u \in \mathrm{dom}(I_h^{cr})$ and $y_h = \Pi_h^{rt} z \in \mathrm{dom}(-D_h^{rt})$, we conclude that

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla_h \Pi_h^{cr} u - \nabla_h u_h^{cr}|^2 \,\mathrm{d}x - \int_\Omega (f_h + \mathrm{div}\, z_h^{rt})\, \Pi_h \Pi_h^{cr} u \,\mathrm{d}x \\ &\qquad + \frac{1}{2}\int_\Omega |\Pi_h \Pi_h^{rt} z - \Pi_h z_h^{rt}|^2 \,\mathrm{d}x - \int_\Omega (f + \mathrm{div}\, z)\, \Pi_h u_h^{cr} \,\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega |\nabla_h \Pi_h^{cr} u - \Pi_h \Pi_h^{rt} z|^2 \,\mathrm{d}x - \int_\Omega (f_h + \mathrm{div}\,\Pi_h^{rt} z)\, \Pi_h \Pi_h^{cr} u \,\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega |\Pi_h z - \Pi_h \Pi_h^{rt} z|^2 \,\mathrm{d}x + \int_\Omega (f + \mathrm{div}\, z)\,(u - \Pi_h \Pi_h^{cr} u) \,\mathrm{d}x\,. \end{aligned}$$

Examples: Obstacle problem

- On the one hand, it holds that

$$\frac{1}{2}\int_\Omega |\Pi_h z - \Pi_h \Pi_h^{rt} z|^2 \,\mathrm{d}x \lesssim h^{2s}\,[z]_{s,2}^2\,.$$

- On the other hand, it holds that

$$\begin{aligned}\int_\Omega (f+\operatorname{div} z)\,(u-\Pi_h\Pi_h^{cr}u)\,\mathrm{d}x &= \int_\Omega (f+\operatorname{div} z)\,(u-\Pi_h^{cr}u)\,\mathrm{d}x \\ &\quad + \int_\Omega (f+\operatorname{div} z)\,(\Pi_h^{cr}u-\Pi_h\Pi_h^{cr}u)\,\mathrm{d}x\,,\end{aligned}$$

  where

  - It holds that

$$\int_\Omega (f+\operatorname{div} z)\,(u-\Pi_h^{cr}u)\,\mathrm{d}x \lesssim \bigg(\int_\Omega |u-\Pi_h^{cr}u|^2\,\mathrm{d}x\bigg)^{\frac{1}{2}} \lesssim h^{2s}\,[z]_{s,2}\,.$$

  - Due to $\Pi_h^{cr}u - \Pi_h\Pi_h^{cr}u = \nabla_h\Pi_h^{cr}u\cdot(\mathrm{id}_{\mathbb{R}^d}-\Pi_h\mathrm{id}_{\mathbb{R}^d})$ and $u=0$ on $\{\operatorname{div} z < -f\}$, it holds that

$$\begin{aligned}-\int_\Omega (f+\operatorname{div} z)\,(\Pi_h^{cr}u-\Pi_h\Pi_h^{cr}u)\,\mathrm{d}x &= \int_\Omega (f+\operatorname{div} z)\,(\nabla_h\Pi_h^{cr}u-\nabla u)\cdot(\mathrm{id}_{\mathbb{R}^d}-\Pi_h\mathrm{id}_{\mathbb{R}^d})\,\mathrm{d}x \\ &\lesssim h\bigg(\int_\Omega |\nabla_h\Pi_h^{cr}u-\nabla u|^2\,\mathrm{d}x\bigg)^{\frac{1}{2}} \lesssim h^{s+1}\,[z]_{s,2}\,.\end{aligned}$$

■

Examples: Obstacle problem

- **Example with central singularity:**

  Let $\Omega := (-1,1)^2$, $\Gamma_D = \partial\Omega$, and $f \in L^2(\Omega)$ s.t. $u \in W^{1,2}(\Omega)$ is given via

$$u(x) := \begin{cases} \frac{|x|^2-2}{2} - \log|x| - \frac{1}{2} & \text{if } |x| > 1\,, \\ 0 & \text{if } |x| \leq 1\,, \end{cases}$$

  so that $z = \nabla u \in (W^{1,2}(\Omega))^2$.

$\Rightarrow$ **Expected for uniform mesh-refinement (*i.e.*, $\theta = 1$):** $\eta^2_{\mathrm{gap},h_k}(\Pi^{cr}_{h_k} u, \Pi^{rt}_{h_k} z) \sim h_k^2 \sim N_k^{-1}$.

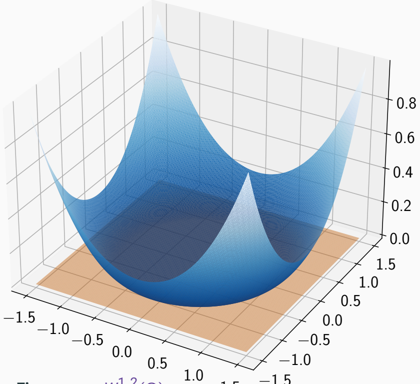


Figure: $u \in W^{1,2}(\Omega)$.

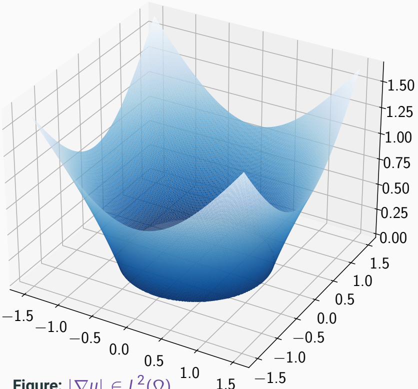
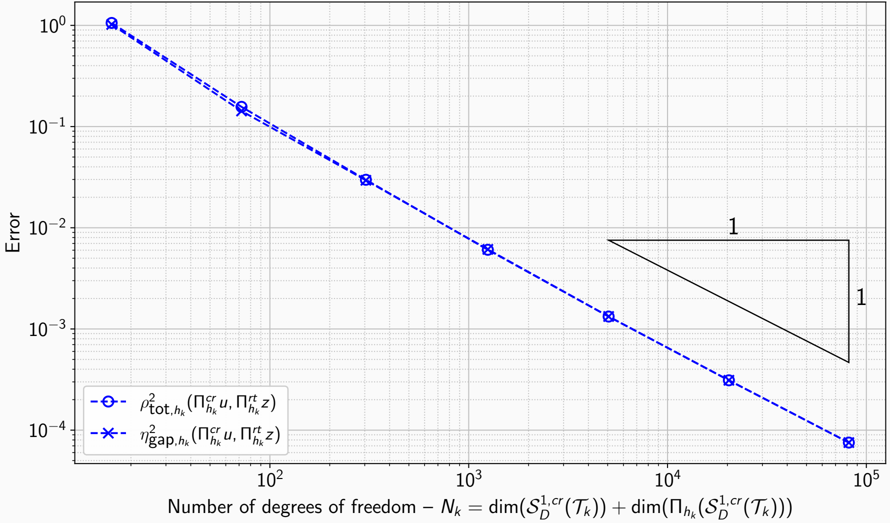


Figure: $|\nabla u| \in L^2(\Omega)$.

- **Experiment:**
  - **Results:**



Conclusions

- ***A posteriori* error analysis:** (on the basis of convex duality)
  - *Primal-dual gap identity*: for every $v \in \mathrm{dom}(I)$ and $y \in \mathrm{dom}(-D)$, it holds that

$$\rho^2_{\mathrm{tot}}(v, y) = \eta^2_{\mathrm{gap}}(v, y)\,.$$

  - *Advantages/Challenges*:

$$\left.\begin{array}{ll} \checkmark \text{ systematic;} & \checkmark \text{ widely applicable;} \\ \checkmark \text{ simple;} & \checkmark \text{ identical;} \end{array}\right\} \quad \text{vs.} \quad \textbf{?}\ \text{numerically practicable.}$$

- ***A posteriori* error analysis:** (on the basis of discrete convex duality)
  - *Discrete reconstruction formulas*: imply numerical practicability.

$$z_h^{rt} = D_t\phi_h(\cdot, \nabla_h u_h^{cr}) + \frac{D_t\psi_h(\cdot, \Pi_h u_h^{cr})}{d}(\mathrm{id}_{\mathbb{R}^d} - \Pi_h \mathrm{id}_{\mathbb{R}^d})\,,$$
$$u_h^{cr} = D_t\psi_h^*(\cdot, \operatorname{div} z_h^{rt}) + D_t\phi_h^*(\cdot, \Pi_h z_h^{rt}) \cdot (\mathrm{id}_{\mathbb{R}^d} - \Pi_h \mathrm{id}_{\mathbb{R}^d})\,.$$

  - *Discrete primal-dual gap identity*: leads to a priori error identity, *i.e.*,

$$\rho^2_{\mathrm{tot},h}(\Pi_h^{cr} u, \Pi_h^{rt} z) = \eta^2_{\mathrm{gap},h}(\Pi_h^{cr} u, \Pi_h^{rt} z)\,.$$

Thank You for today!

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