A priori and *a posteriori* error identities for convex minimization problems based on convex duality relations

Lecture 3

Alex Kaltenbach

"Oberseminar" of the Department of Applied Mathematics,
University of Freiburg,
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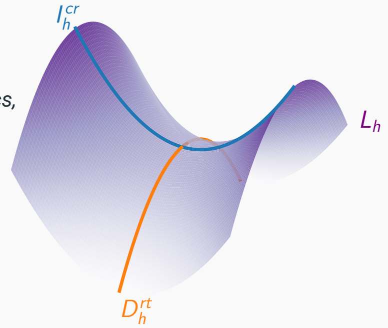


Content of the Lecture 3

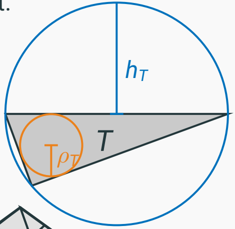
- **Lecture 3: Convex duality theory for discrete integral functionals**
  - Crouzeix–Raviart element and Raviart–Thomas element;
    - Triangulations and discrete spaces;
    - Crouzeix–Raviart element and special features;
    - Raviart–Thomas element and special features;
    - Relations.
  - Fenchel duality theory for discrete integral functionals;
    - Integral representation of discrete dual energy functional;
    - Discrete Fenchel duality relations;
    - Discrete reconstruction formulas;
    - Examples.

Crouzeix–Raviart element and Raviart–Thomas element

Triangulation

- **Triangulation:** Let $\{\mathcal{T}_h\}_{h>0}$ be *shape-regular* triangulations of the *simplicial Lipschitz domain* Ω, *i.e.*, there exists a constant $\omega_0 > 0$ s.t.

$$\sup_{T\in\mathcal{T}_h} \frac{h_T}{\rho_T} \leq \omega_0\,,$$



where, for every $T \in \mathcal{T}_h$, we denote by

- $\rho_T \coloneqq \sup\{r > 0 \mid \exists x \in T \,:\, B_r^d(x) \subset T\}$;
- $h_T \coloneqq \mathrm{diam}(T)$.

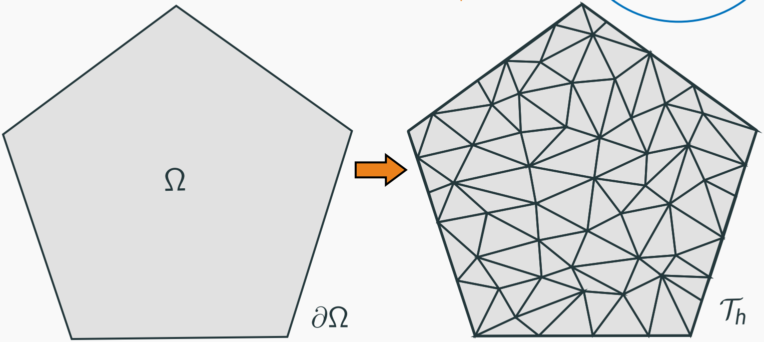
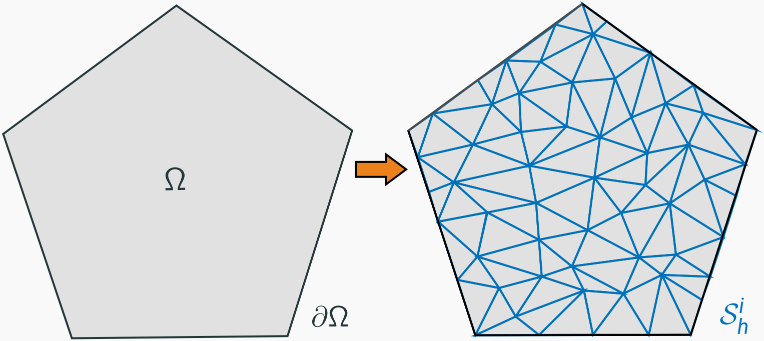


Figure: Triangulation of pentagon $\Omega \subseteq \mathbb{R}^2$.

Sides

- **Sets of sides:**

$$\mathcal{S}_h^i \coloneqq \{T \cap T' \mid T, T' \in \mathcal{T}_h : \dim_{\mathscr{H}}(T \cap T') = d-1\}\,,$$



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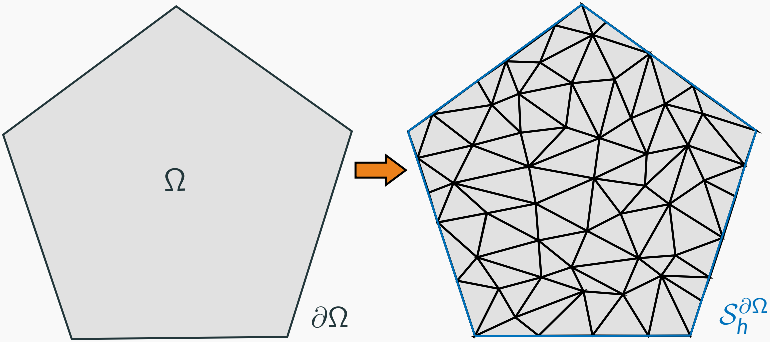

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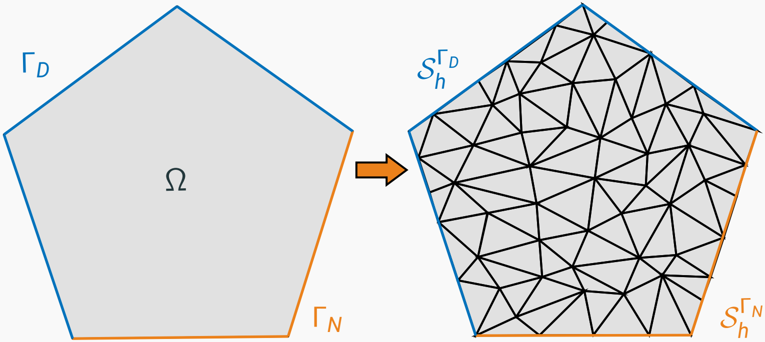
$$\mathcal{S}_h^{\Gamma_D} \coloneqq \{S \in \mathcal{S}_h^{\partial\Omega} \mid \operatorname{int}(S) \subseteq \Gamma_D\}\,,$$
$$\mathcal{S}_h^{\Gamma_N} \coloneqq \{S \in \mathcal{S}_h^{\partial\Omega} \mid \operatorname{int}(S) \subseteq \Gamma_N\}\,,$$



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$$\mathcal{S}_h^{\Gamma_N} \coloneqq \{S \in \mathcal{S}_h^{\partial\Omega} \mid \operatorname{int}(S) \subseteq \Gamma_N\}\,,$$
$$\mathcal{S}_h \coloneqq \mathcal{S}_h^{i} \cup \mathcal{S}_h^{\partial\Omega}\,.$$

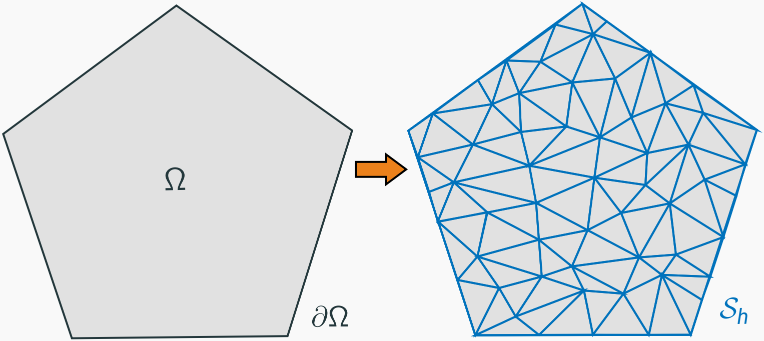

Figure: Triangulation of pentagon $\Omega \subseteq \mathbb{R}^2$.

Discrete spaces

- **Discrete spaces:** For *polynomial degree* $k \in \mathbb{N} \cup \{0\}$, we define

$$\mathbb{P}^k(\mathcal{T}_h) \coloneqq \left\{ v_h \in L^\infty(\Omega) \mid v_h|_T \in P^k(T) \text{ for all } T \in \mathcal{T}_h \right\},$$

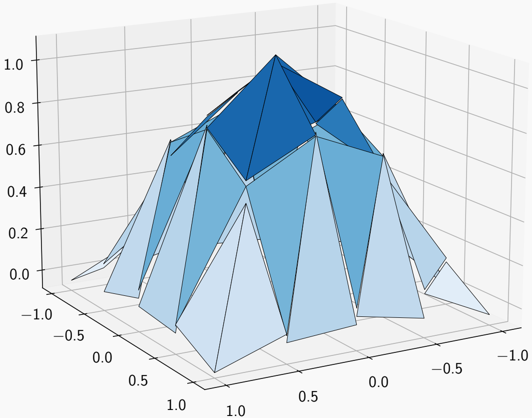

Figure: (local) L^2-projection onto $\mathbb{P}^k(\mathcal{T}_h)$ of $u(x_1, x_2) = (1 - x_1^2)(1 - x_2^2)$.

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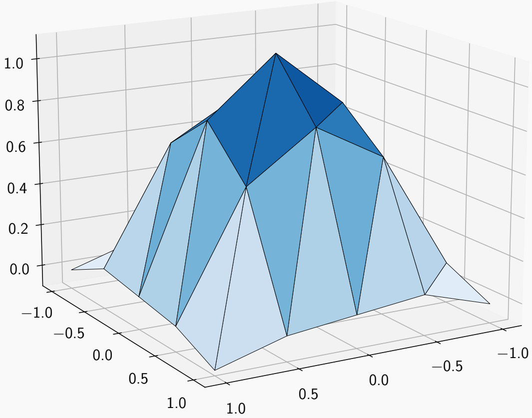
$$\mathcal{S}^k(\mathcal{T}_h) \coloneqq W^{1,1}(\Omega) \cap \mathbb{P}^k(\mathcal{T}_h),$$



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$$\mathcal{S}^k(\mathcal{T}_h) \coloneqq W^{1,1}(\Omega) \cap \mathbb{P}^k(\mathcal{T}_h),$$

$$\mathcal{S}^k_D(\mathcal{T}_h) \coloneqq W^{1,1}_D(\Omega) \cap \mathbb{P}^k(\mathcal{T}_h).$$

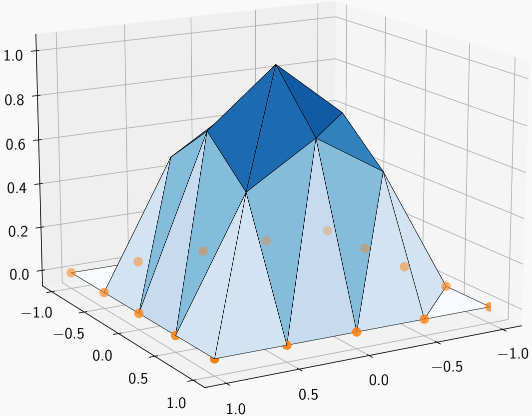
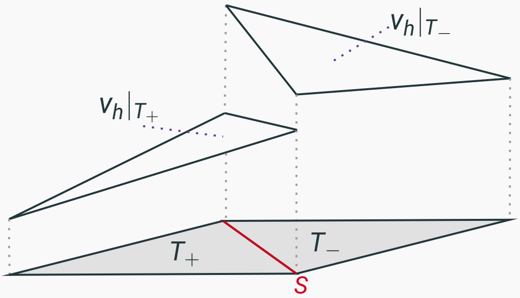


Figure: nodal interpolation into $\mathcal{S}^k_D(\mathcal{T}_h)$ of $u(x_1, x_2) = (1 - x_1^2)(1 - x_2^2)$.

Jumps and averages



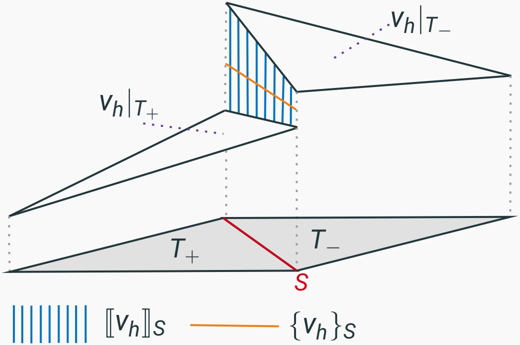
Jumps and averages

- **Jumps**: For every $v_h \in \mathbb{P}^k(\mathcal{T}_h)$ and $S \in \mathcal{S}_h$, we define

$$[\![v_h]\!]_S \coloneqq \begin{cases} v_h|_{T_+} - v_h|_{T_-} & \text{if } S \in \mathcal{S}_h^{i}\,,\ T_+, T_- \in \mathcal{T}_h \text{ s.t. } \partial T_+ \cap \partial T_- = S\,, \\ v_h|_T & \text{if } S \in \mathcal{S}_h^{\partial\Omega}\,,\ T \in \mathcal{T}_h \text{ s.t. } S \subseteq \partial T\,. \end{cases}$$

- **Averages**: For every $v_h \in \mathbb{P}^k(\mathcal{T}_h)$ and $S \in \mathcal{S}_h$, we define

$$\{v_h\}_S \coloneqq \begin{cases} \frac{1}{2}(v_h|_{T_+} + v_h|_{T_-}) & \text{if } S \in \mathcal{S}_h^{i}\,,\ T_+, T_- \in \mathcal{T}_h \text{ s.t. } \partial T_+ \cap \partial T_- = S\,, \\ v_h|_T & \text{if } S \in \mathcal{S}_h^{\partial\Omega}\,,\ T \in \mathcal{T}_h \text{ s.t. } S \subseteq \partial T\,. \end{cases}$$



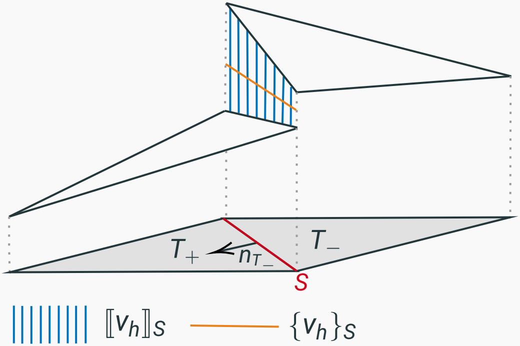
Normal jumps and normal averages

- **Normal jumps:** For every $y_h \in (\mathbb{P}^k(\mathcal{T}_h))^d$ and $S \in \mathcal{S}_h$, we define

$$[\![y_h \cdot n]\!]_S \coloneqq \begin{cases} y_h|_{T_+} \cdot n_{T_+} + y_h|_{T_-} \cdot n_{T_-} & \text{if } S \in \mathcal{S}_h^i\,,\ T_+, T_- \in \mathcal{T}_h \text{ s.t. } \partial T_+ \cap \partial T_- = S\,, \\ y_h|_T \cdot n & \text{if } S \in \mathcal{S}_h^{\partial\Omega}\,,\ T \in \mathcal{T}_h \text{ s.t. } S \subseteq \partial T\,, \end{cases}$$

- **Normal averages:** For every $y_h \in (\mathbb{P}^k(\mathcal{T}_h))^d$ and $S \in \mathcal{S}_h$, we define

$$\{y_h \cdot n\}_S \coloneqq \begin{cases} \tfrac{1}{2}(y_h|_{T_+} \cdot n_{T_+} - y_h|_{T_-} \cdot n_{T_-}) & \text{if } S \in \mathcal{S}_h^i\,,\ T_+, T_- \in \mathcal{T}_h \text{ s.t. } \partial T_+ \cap \partial T_- = S\,, \\ y_h|_T \cdot n & \text{if } S \in \mathcal{S}_h^{\partial\Omega}\,,\ T \in \mathcal{T}_h \text{ s.t. } S \subseteq \partial T\,, \end{cases}$$



- **Crouzeix–Raviart element:** (*cf.* [2, Crouzeix & Raviart, '73])

$$\mathcal{S}^{1,cr}(\mathcal{T}_h) \coloneqq \left\{ v_h \in \mathbb{P}^1(\mathcal{T}_h) \;\middle|\; \fint_S [\![v_h]\!]_S \,\mathrm{d}s = [\![v_h]\!]_S(x_S) = 0 \text{ for all } S \in \mathcal{S}_h^i \right\},$$

$$\mathcal{S}^{1,cr}_D(\mathcal{T}_h) \coloneqq \left\{ v_h \in \mathcal{S}^{1,cr}(\mathcal{T}_h) \;\middle|\; \fint_S v_h \,\mathrm{d}s = v_h(x_S) = 0 \text{ for all } S \in \mathcal{S}_h^{\Gamma_D} \right\}.$$

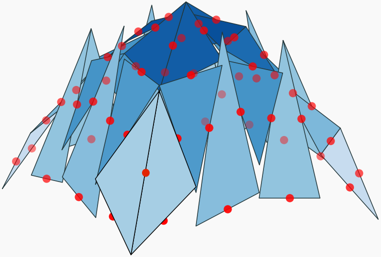


Figure: Crouzeix–Raviart minimizer of Dirichlet energy with $\Omega \coloneqq (-1,1)^2$, $\Gamma_D \coloneqq \partial\Omega$ and $f \coloneqq 1$.

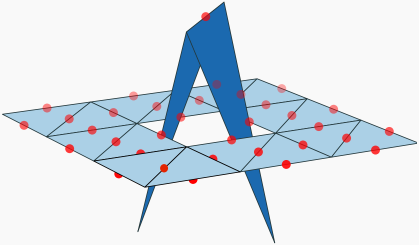


Figure: Crouzeix–Raviart basis function.

- **(Non-)Conformity:** $\mathcal{S}^{1,cr}(\mathcal{T}_h) \not\subseteq W^{1,p}(\Omega)$.

$x_S \coloneqq \frac{1}{d}\sum_{\nu \in \mathcal{N}_h\,:\,\nu \in T} \nu$ for all $S \in \mathcal{S}_h$.

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$$\mathcal{S}_D^{1,cr}(\mathcal{T}_h) \coloneqq \left\{ v_h \in \mathcal{S}^{1,cr}(\mathcal{T}_h) \;\middle|\; \fint_S v_h \,\mathrm{d}s = v_h(x_S) = 0 \text{ for all } S \in \mathcal{S}_h^{\Gamma_D} \right\}.$$

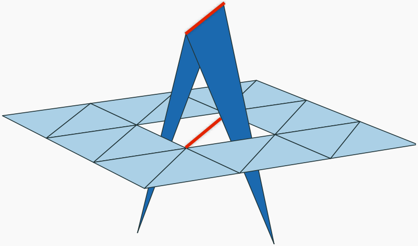
- **Basis functions:** $(\varphi_S)_{S\in\mathcal{S}_h} \subseteq \mathcal{S}^{1,cr}(\mathcal{T}_h)$ s.t.

$$\varphi_S(x_{S'}) = \delta_{SS'} \quad \text{for all } S, S' \in \mathcal{S}_h\,,$$

e.g., for $T \in \mathcal{T}_h$ s.t. $S \subseteq \partial T$,

$$\varphi_S \coloneqq 1 - d\varphi_{\nu_S} \quad \text{in } T\,,$$

where $\nu_S \in \mathcal{N}_h$ with $\nu_S \in T$ and $\nu_S \notin S$.



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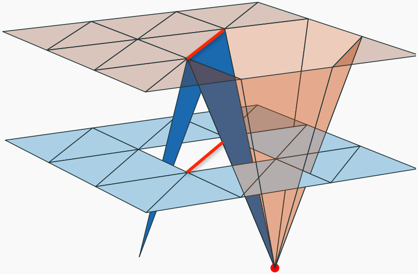

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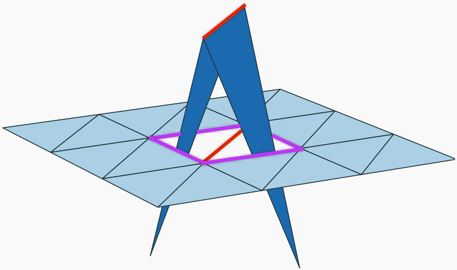
Crouzeix–Raviart element

	Crouzeix–Raviart	Courant
local DOFs	Edges/Facets, *i.e.*, • 3 per element in 2D; • 4 per element in 3D.	Vertices, *i.e.*, • 3 per element in 2D; • 4 per element in 3D.
global DOFs	$\mathrm{card}(\mathcal{S}_h)$ ($\approx 3 \times \mathrm{card}(\mathcal{N}_h)$ in 2D)	$\mathrm{card}(\mathcal{N}_h)$
Parallelization	$\mathrm{supp}(\varphi_S) \subseteq \omega_S$, *i.e.*, on 2 elements	$\mathrm{supp}(\varphi_\nu) \subseteq \omega_\nu$, *i.e.*, on vertex patch
Duality	conforming dual problem	non-conforming dual problem

Figure: Comparison of Crouzeix–Raviart and Courant element in 2D and 3D.



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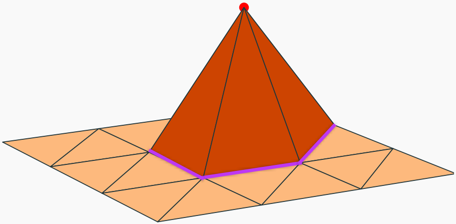


Figure: Courant basis function.

Special features of the Crouzeix–Raviart element

- **Fortin interpolation operator:** $\Pi_h^{cr}: W_D^{1,p}(\Omega) \to \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, for every $v \in W_D^{1,p}(\Omega)$, defined by

$$\Pi_h^{cr} v \coloneqq \sum_{S \in \mathcal{S}_h} \langle v \rangle_S \, \varphi_S \,, \quad \text{where} \quad \langle v \rangle_S \coloneqq \fint_S v \, \mathrm{d}s \quad \text{for all } S \in \mathcal{S}_h \,,$$

preserves averages of gradients and moments (on sides), *i.e.*, for every $v \in W_D^{1,p}(\Omega)$, it holds that

$$\begin{aligned} \nabla_h \Pi_h^{cr} v &= \Pi_h \nabla v && \text{in } (\mathbb{P}^0(\mathcal{T}_h))^d \,, \\ \pi_h \Pi_h^{cr} v &= \pi_h v && \text{in } \mathbb{P}^0(\mathcal{S}_h) \,, \end{aligned}$$

where

- $\Pi_h : L^1(\Omega) \to \mathbb{P}^0(\mathcal{T}_h)$, for every $v \in L^1(\Omega)$, is defined by

$$\Pi_h v \coloneqq \sum_{T \in \mathcal{T}_h} \langle v \rangle_T \chi_T \,, \quad \text{where} \quad \langle v \rangle_T \coloneqq \fint_T v \, \mathrm{d}s \quad \text{for all } T \in \mathcal{T}_h \,;$$

- $\pi_h : L^1(\cup \mathcal{S}_h) \to \mathbb{P}^0(\mathcal{S}_h)$, for every $v \in L^1(\cup \mathcal{S}_h)$, is defined by

$$\pi_h v \coloneqq \sum_{S \in \mathcal{S}_h} \langle v \rangle_S \chi_S \,, \quad \text{where} \quad \langle v \rangle_S \coloneqq \fint_S v \, \mathrm{d}s \quad \text{for all } S \in \mathcal{S}_h \,.$$

Theorem: (stability with constant 1)

If $\phi\colon \mathbb{R}^d \to \mathbb{R}$ is convex, then for every $v \in W^{1,p}_D(\Omega)$, it holds that

$$\int_\Omega \phi(\nabla_h \Pi_h^{cr} v)\,\mathrm{d}x \leq \int_\Omega \phi(\nabla v)\,\mathrm{d}x\,,$$

i.e., with constant 1.

Proof.

- By *Jensen's inequality*, for every $T \in \mathcal{T}_h$, it holds that

$$\phi\Big(\fint_T \nabla v\,\mathrm{d}y\Big) \leq \fint_T \phi(\nabla v)\,\mathrm{d}y\,.$$

- Due to $\nabla_h \Pi_h^{cr} v = \Pi_h \nabla v$, we conclude that

$$\begin{aligned}\int_\Omega \phi(\nabla_h \Pi_h^{cr} v)\,\mathrm{d}x &= \sum_{T \in \mathcal{T}_h} \int_T \phi\Big(\fint_T \nabla v\,\mathrm{d}y\Big)\,\mathrm{d}x \\ &\leq \sum_{T \in \mathcal{T}_h} \Big(\int_T 1\,\mathrm{d}x\Big) \fint_T \phi(\nabla v)\,\mathrm{d}y \\ &= \int_\Omega \phi(\nabla v)\,\mathrm{d}x\,.\end{aligned}$$

■

Discrete Poincaré inequality

Theorem: (discrete Poincaré inequality)

If $|\Gamma_D| > 0$, then for every $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$, it holds that

$$\int_\Omega |v_h|^p \,\mathrm{d}x \lesssim_h \int_\Omega |\nabla_h v_h|^p \,\mathrm{d}x\,.$$

Proof (by sketch). Show that $\ker(\nabla_h|_{\mathcal{S}^{1,cr}_D(\mathcal{T}_h)}) = \{0\}$.

- Let $v_h \in \ker(\nabla_h|_{\mathcal{S}^{1,cr}_D(\mathcal{T}_h)})$, i.e., for every $T \in \mathcal{T}_h$, it holds that

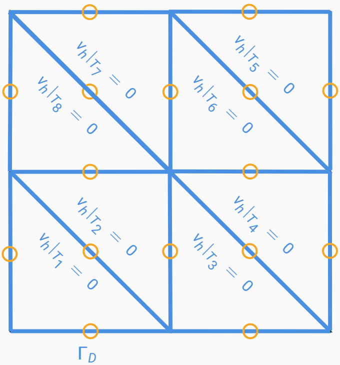
$$\nabla(v_h|_T) = 0 \quad \text{in } T\,.$$

→ For every $T \in \mathcal{T}_h$, there exists $c_T \in \mathbb{R}$ s.t.

$$v_h|_T = c_T \quad \text{in } T\,.$$

→ Due to $v_h(x_S) = 0$ for every $S \in \mathcal{S}_h^{\Gamma_D} \neq \emptyset$, it follows that

$$v_h = 0 \quad \text{a.e. in } \Omega\,. \qquad \blacksquare$$



Raviart–Thomas element

- **Raviart–Thomas element:** (*cf.* [4, Raviart & Thomas, '77])

$$\mathcal{RT}^0(\mathcal{T}_h) \coloneqq \{y_h \in (\mathbb{P}^1(\mathcal{T}_h))^d \mid y_h|_T \cdot n_T = \mathrm{const} \text{ on } \partial T \text{ for all } T \in \mathcal{T}_h\,,\ [\![y_h \cdot n]\!]_S = 0 \text{ on } S \text{ for all } S \in \mathcal{S}_h^i\}\,,$$

$$\mathcal{RT}^0_N(\mathcal{T}_h) \coloneqq \{y_h \in \mathcal{RT}^0(\mathcal{T}_h) \mid y_h \cdot n = 0 \text{ a.e. on } \Gamma_N\}\,.$$

- **Basis functions:** $(\psi_S)_{S\in\mathcal{S}_h} \subseteq \mathcal{RT}^0(\mathcal{T}_h)$ s.t.

$$\psi_S \cdot n_{S'} = \delta_{SS'} \quad \text{on } S' \quad \text{for all } S, S' \in \mathcal{S}_h\,,$$

e.g.,

$$\psi_S(x) \coloneqq \begin{cases} \pm\frac{|S|}{(d!)|T_\pm|}(\nu_\pm - x) & \text{if } x \in T_\pm\,,\\ 0 & \text{if } x \in \Omega \setminus (T_+ \cup T_-)\,. \end{cases}$$

- **Conformity:** $\mathcal{RT}^0(\mathcal{T}_h) \subseteq W^{p'}(\mathrm{div};\Omega)$.

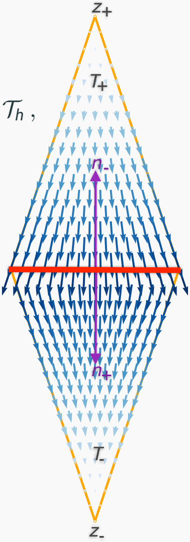


Figure: Raviart–Thomas basis function.

Special features of the Raviart–Thomas element

- **Fortin interpolation operator:** $\Pi_h^{rt}\colon W_N^{p'}(\mathrm{div};\Omega)\cap(W^{1,1}(\Omega))^d\to\mathcal{RT}_N^0(\mathcal{T}_h)$, for every $y\in W_N^{p'}(\mathrm{div};\Omega)\cap(W^{1,1}(\Omega))^d$ defined by

$$\Pi_h^{rt}y \coloneqq \sum_{S\in\mathcal{S}_h}\langle y\cdot n\rangle_S\,\psi_S\,,\quad\text{where}\quad \langle y\cdot n\rangle_S\coloneqq \fint_S y\cdot n\,\mathrm{d}s\quad\text{for all } S\in\mathcal{S}_h\,,$$

preserves averages of divergences and moments of normal traces (on sides), *i.e.*, for every $y\in W_N^{p'}(\mathrm{div};\Omega)\cap(W^{1,1}(\Omega))^d$, it holds that

$$\begin{aligned}\mathrm{div}\,\Pi_h^{rt}y &= \Pi_h\mathrm{div}\,y &&\text{in } \mathbb{P}^0(\mathcal{T}_h)\,,\\ \pi_h[\Pi_h^{rt}y\cdot n] &= \pi_h[y\cdot n] &&\text{in } \mathbb{P}^0(\mathcal{S}_h)\,,\end{aligned}$$

where

- $\Pi_h\colon L^1(\Omega)\to\mathbb{P}^0(\mathcal{T}_h)$, for every $v\in L^1(\Omega)$, is defined by

$$\Pi_h v\coloneqq\sum_{T\in\mathcal{T}_h}\langle v\rangle_T\chi_T\,,\quad\text{where}\quad\langle v\rangle_T\coloneqq\fint_T v\,\mathrm{d}s\quad\text{for all } T\in\mathcal{T}_h\,;$$

- $\pi_h\colon L^1(\cup\mathcal{S}_h)\to\mathbb{P}^0(\mathcal{S}_h)$, for every $v\in L^1(\cup\mathcal{S}_h)$, is defined by

$$\pi_h v\coloneqq\sum_{S\in\mathcal{S}_h}\langle v\rangle_S\chi_S\,,\quad\text{where}\quad\langle v\rangle_S\coloneqq\fint_S v\,\mathrm{d}s\quad\text{for all } S\in\mathcal{S}_h\,.$$

Key ingredient I: discrete surjectivity of divergence operator

Lemma: (key ingredient I: discrete surjectivity of divergence operator)

The following statements apply:

(i) If $\Gamma_N \neq \partial\Omega$, then $\operatorname{div}\colon \mathcal{RT}^0_N(\mathcal{T}_h) \to \mathbb{P}^0(\mathcal{T}_h)$ is surjective;

(ii) If $\Gamma_N = \partial\Omega$, then $\operatorname{div}\colon \mathcal{RT}^0_N(\mathcal{T}_h) \to \mathbb{P}^0_0(\mathcal{T}_h) := \mathbb{P}^0(\mathcal{T}_h)/\mathbb{R}$ is surjective.

Proof. If $p<2$, $\Pi_h^{rt}\colon W^{p'}_N(\operatorname{div};\Omega)\to\mathcal{RT}^0_N(\mathcal{T}_h)$ is still well-defined (*cf.* [3, Ern, Guermond, '21]).

ad (i). Since $\operatorname{div}(W^{p'}_N(\operatorname{div};\Omega))=L^{p'}(\Omega)$, for every $f_h\in\mathbb{P}^0(\mathcal{T}_h)$, there is $y\in W^{p'}_N(\operatorname{div};\Omega)$ s.t.

$$\operatorname{div} y = f_h \quad \text{a.e. in } \Omega\,.$$

Then, $y_h := \Pi_h^{rt} y \in \mathcal{RT}^0_N(\mathcal{T}_h)$ satisfies

$$\left.\begin{aligned}\operatorname{div} y_h &= \Pi_h \operatorname{div} y\\ &= \Pi_h f_h = f_h\end{aligned}\right\}\quad \text{a.e. in } \Omega\,.$$

ad (ii). Since $\operatorname{div}(W^{p'}_N(\operatorname{div};\Omega))=L^{p'}_0(\Omega)$, for every $f_h\in\mathbb{P}^0_0(\mathcal{T}_h)$, there is $y\in W^{p'}_N(\operatorname{div};\Omega)$ s.t.

$$\operatorname{div} y = f_h \quad \text{a.e. in } \Omega\,.$$

Then, $y_h := \Pi_h^{rt} y \in \mathcal{RT}^0_N(\mathcal{T}_h)$ satisfies

$$\left.\begin{aligned}\operatorname{div} y_h &= \Pi_h \operatorname{div} y\\ &= \Pi_h f_h = f_h\end{aligned}\right\}\quad \text{a.e. in } \Omega\,.$$

■

Discrete integration-by-parts formula

Lemma: (discrete integration-by-parts formula)

For every $v_h \in \mathcal{S}^{1,cr}(\mathcal{T}_h)$ and $y_h \in \mathcal{RT}^0(\mathcal{T}_h)$, it holds that

$$\int_\Omega \nabla_h v_h \cdot y_h \,\mathrm{d}x + \int_\Omega v_h \operatorname{div} y_h \,\mathrm{d}x = \int_{\partial\Omega} v_h\, y_h \cdot n \,\mathrm{d}s \,.$$

Proof.

- Element-wise integration-by-parts yields that

$$\begin{aligned}
\int_\Omega \nabla_h v_h \cdot y_h \,\mathrm{d}x + \int_\Omega v_h \operatorname{div} y_h \,\mathrm{d}x &= \sum_{T\in\mathcal{T}_h} \Big[\int_T \nabla_h v_h \cdot y_h \,\mathrm{d}x + \int_T v_h \operatorname{div} y_h \,\mathrm{d}x\Big] \\
&= \sum_{T\in\mathcal{T}_h} \int_{\partial T} v_h\, y_h \cdot n \,\mathrm{d}s \\
&= \int_{\partial\Omega} v_h\, y_h \cdot n \,\mathrm{d}s + \sum_{S\in\mathcal{S}_h^i} \int_S [\![v_h y_h \cdot n]\!]_S \,\mathrm{d}s \,.
\end{aligned}$$

- The product rule $[\![v_h y_h \cdot n]\!]_S = [\![v_h]\!]_S \{y_h \cdot n\}_S + \{v_h\}_S [\![y_h \cdot n]\!]_S$ for $S \in \mathcal{S}_h^i$ yields that

$$\sum_{S\in\mathcal{S}_h^i} \int_S [\![v_h y_h \cdot n]\!]_S \,\mathrm{d}s = \sum_{S\in\mathcal{S}_h^i} \Big[\{y_h \cdot n\}_S \underbrace{\int_S [\![v_h]\!]_S \,\mathrm{d}s}_{=0} + \underbrace{[\![y_h \cdot n]\!]_S}_{=0} \int_S \{v_h\}_S \,\mathrm{d}s\Big] = 0 \,. \quad \blacksquare$$

Lemma: (key ingredient II: discrete orthogonality relation)

$$\ker(\mathrm{div}|_{\mathcal{RT}^0_N(\mathcal{T}_h)}) = (\nabla_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h)))^\perp \qquad \text{(in } (\mathbb{P}^0(\mathcal{T}_h))^d).$$

Proof.

ad '⊆'. For $y_h \in \ker(\mathrm{div}|_{\mathcal{RT}^0_N(\mathcal{T}_h)})$, for every $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$, it holds that

$$\int_\Omega \nabla_h v_h \cdot y_h \,\mathrm{d}x = -\int_\Omega v_h \underbrace{\mathrm{div}\, y_h}_{=\,0} \,\mathrm{d}x = 0\,,$$

i.e., $y_h \in (\nabla_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h)))^\perp$.

ad '⊇'. For $y_h \in (\nabla_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h)))^\perp$, for every $S \in \mathcal{S}^i_h$, it holds that

$$0 = \int_\Omega \nabla_h \varphi_S \cdot y_h \,\mathrm{d}x = [\![y_h \cdot n]\!]_S\, |S|\,,$$

i.e., $y_h \in \mathcal{RT}^0(\mathcal{T}_h)$ with $\mathrm{div}\, y_h = 0$ a.e. in Ω.

→ For every $S \in \mathcal{S}^{\Gamma_N}_h$, it holds that

$$0 = \int_\Omega \nabla_h \varphi_S \cdot y_h \,\mathrm{d}x = y_h \cdot n|_S\, |S|\,,$$

i.e., $y_h \in \ker(\mathrm{div}|_{\mathcal{RT}^0_N(\mathcal{T}_h)})$. ■

Fenchel duality theory
for discrete integral functionals

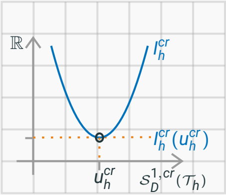
Discrete primal problem

- **Three non-conforming modifications:**
  1. Replace ϕ and ψ by *element-wise approximations* ϕ_h and ψ_h, *i.e.*,
     - $\phi_h\colon \Omega\times\mathbb{R}^d\to\mathbb{R}\cup\{+\infty\}$ is measurable s.t. $\phi_h(x,\cdot)\in\Gamma_0(\mathbb{R}^d)$ for a.e. $x\in\Omega$;
     - $\psi_h\colon \Omega\times\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ is measurable s.t. $\psi_h(x,\cdot)\in\Gamma_0(\mathbb{R}^d)$ for a.e. $x\in\Omega$;
     - $\phi_h(\cdot,t),\psi_h(\cdot,s)\in\mathbb{P}^0(\mathcal{T}_h)$ for all $t\in\mathbb{R}^d$ and $s\in\mathbb{R}$.
  2. *(Local) L^2-projection operator* $\Pi_h\colon L^1(\Omega)\to\mathbb{P}^0(\mathcal{T}_h)$;
  3. *Element-wise gradient operator* $\nabla_h\colon \mathcal{S}^{1,cr}_D(\mathcal{T}_h)\to(\mathbb{P}^0(\mathcal{T}_h))^d$.

- **Discrete primal problem:** Min. $I_h^{cr}\colon\mathcal{S}^{1,cr}_D(\mathcal{T}_h)\to\mathbb{R}\cup\{+\infty\}$, for every $v_h\in\mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ defined by

$$I_h^{cr}(v_h)\coloneqq\int_\Omega\phi_h(\cdot,\nabla_h v_h)\,\mathrm{d}x+\int_\Omega\psi_h(\cdot,\Pi_h v_h)\,\mathrm{d}x\,.$$

- **Assumption:** A minimizer $u_h^{cr}\in\mathcal{S}^{1,cr}_D(\mathcal{T}_h)$, a so-called *discrete primal solution*, exists.



Discrete (Fenchel) primal problem

- **Setup of a discrete (Fenchel) primal problem:**
  - Let $G_h\colon (\mathbb{P}^0(\mathcal{T}_h))^d \to \mathbb{R}\cup\{+\infty\}$, for every $y_h \in (\mathbb{P}^0(\mathcal{T}_h))^d$, be defined by

$$G_h(y_h) \coloneqq \int_\Omega \phi_h(\cdot, y_h)\,\mathrm{d}x\,.$$

→ $G_h \in \Gamma_0((\mathbb{P}^0(\mathcal{T}_h))^d)$;

  - Let $F_h\colon \mathcal{S}_D^{1,cr}(\mathcal{T}_h) \to \mathbb{R}\cup\{+\infty\}$, for every $v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, be defined by

$$F_h(v_h) \coloneqq \int_\Omega \psi_h(\cdot, \Pi_h v_h)\,\mathrm{d}x\,.$$

→ $F_h \in \Gamma_0(\mathcal{S}_D^{1,cr}(\mathcal{T}_h))$;

  - Let $\Lambda_h\colon \mathcal{S}_D^{1,cr}(\mathcal{T}_h) \to (\mathbb{P}^0(\mathcal{T}_h))^d$, for every $v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, be defined by

$$\Lambda_h v_h \coloneqq \nabla_h v_h\,.$$

→ $\Lambda_h \in L(\mathcal{S}_D^{1,cr}(\mathcal{T}_h); (\mathbb{P}^0(\mathcal{T}_h))^d)$.

→ **Discrete (Fenchel) primal problem:** For every $v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, we have that

$$I_h^{cr}(v_h) = G_h(\Lambda_h v_h) + F_h(v_h)\,.$$

- **Discrete (Fenchel) dual problem:** Maximize $D_h^0 : (\mathbb{P}^0(\mathcal{T}_h))^d \to \mathbb{R} \cup \{-\infty\}$, for every $y_h \in (\mathbb{P}^0(\mathcal{T}_h))^d$, defined by

$$D_h^0(y_h) \coloneqq -F_h^*(-\Lambda_h^* y_h) - G_h^*(y_h)\,,$$

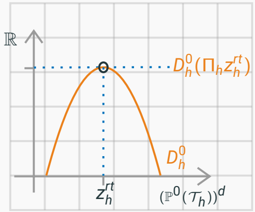
where

  - For every $y_h \in (\mathbb{P}^0(\mathcal{T}_h))^d$, it holds that

$$\begin{aligned} G_h^*(y_h) &= \sup_{\widehat{y}_h \in (\mathbb{P}^0(\mathcal{T}_h))^d} \bigg\{ \int_\Omega y_h \cdot \widehat{y}_h \,\mathrm{d}x - \int_\Omega \phi_h(\cdot, \widehat{y}_h)\,\mathrm{d}x \bigg\} \\ &= \sum_{T \in \mathcal{T}_h} \int_T \sup_{t \in \mathbb{R}^d} \{ y_h(x_T) \cdot t - \phi_h(x_T, t) \}\,\mathrm{d}x \\ &= \int_\Omega \phi_h^*(\cdot, y_h)\,\mathrm{d}x\,; \end{aligned}$$

  - For every $y_h \in (\mathbb{P}^0(\mathcal{T}_h))^d$, we have that

$$F_h^*(-\Lambda_h^* y_h) = \sup_{v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)} \bigg\{ \int_\Omega -y_h \cdot \nabla_h v_h \,\mathrm{d}x - \int_\Omega \psi_h(\cdot, \Pi_h v_h)\,\mathrm{d}x \bigg\}\,.$$



Integral representation of dual problem

- **Integral representation of $F_h^* \circ (-\Lambda_h^*)$:** For every $y_h \in \mathcal{R}T_N^0(\mathcal{T}_h)$, we have that

$$F_h^*(-\Lambda_h^*\Pi_h y_h) = \sup_{v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)} \left\{ \int_\Omega -\Pi_h y_h \cdot \nabla_h v_h \,\mathrm{d}x - \int_\Omega \psi_h(\cdot, \Pi_h v_h)\,\mathrm{d}x \right\}$$

$$= \sup_{v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)} \left\{ \int_\Omega \operatorname{div} y_h \,\Pi_h v_h \,\mathrm{d}x - \int_\Omega \psi_h(\cdot, \Pi_h v_h)\,\mathrm{d}x \right\}$$

when? $\quad \left(= \int_\Omega \psi_h^*(\cdot, \operatorname{div} y_h)\,\mathrm{d}x \right).$

Lemma: (Fenchel conjugate of integral functionals defined on $\Pi_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h))$)

Let one the following two assumptions be satisfied:

- $\Gamma_D \neq \partial\Omega$;
- $\psi_h(x,\cdot) \in C^1(\mathbb{R})$ for a.e. $x \in \Omega$.

Then, for every $\hat{v}_h \in \mathbb{P}^0(\mathcal{T}_h)$, it holds that

$$\int_\Omega \psi_h^*(\cdot,\hat{v}_h)\,\mathrm{d}x = \sup_{v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)} \left\{ \int_\Omega \hat{v}_h\,\Pi_h v_h\,\mathrm{d}x - \int_\Omega \psi_h(\cdot,\Pi_h v_h)\,\mathrm{d}x \right\}.$$

Proof (of the case $\Gamma_D \neq \partial\Omega$).

- Appealing to [1, Bartels & Wang, '21], it holds that

$$\mathbb{P}^0(\mathcal{T}_h) = \Pi_h(\mathcal{S}^{1,cr}_D(\mathcal{T}_h)).$$

→ For every $\hat{v}_h \in \mathbb{P}^0(\mathcal{T}_h)$, we find that

$$\sup_{v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)} \left\{ \int_\Omega \hat{v}_h\,\Pi_h v_h\,\mathrm{d}x - \int_\Omega \psi_h(\cdot,\Pi_h v_h)\,\mathrm{d}x \right\} = \sup_{v_h \in \mathbb{P}^0(\mathcal{T}_h)} \left\{ \int_\Omega \hat{v}_h\,v_h\,\mathrm{d}x - \int_\Omega \psi_h(\cdot,v_h)\,\mathrm{d}x \right\}$$

$$= \int_\Omega \psi_h^*(\cdot,\hat{v}_h)\,\mathrm{d}x. \qquad \blacksquare$$

Integral representation of dual problem

- **Integral representation of $F_h^* \circ (-\Lambda_h^*)$**: For every $y_h \in \mathcal{RT}_N^0(\mathcal{T}_h)$, we have that

$$F_h^*(-\Lambda_h^*\Pi_h y_h) = \sup_{v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)} \left\{ \int_\Omega -\Pi_h y_h \cdot \nabla_h v_h \,\mathrm{d}x - \int_\Omega \psi_h(\cdot, \Pi_h v_h)\,\mathrm{d}x \right\}$$

$$= \sup_{v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)} \left\{ \int_\Omega \operatorname{div} y_h \, \Pi_h v_h \,\mathrm{d}x - \int_\Omega \psi_h(\cdot, \Pi_h v_h)\,\mathrm{d}x \right\}$$

when? $$\left(= \int_\Omega \psi_h^*(\cdot, \operatorname{div} y_h)\,\mathrm{d}x \right).$$

- **Assumption**: For every $y_h \in \mathcal{RT}_N^0(\mathcal{T}_h)$, we have that

$$F_h^*(-\Lambda_h^*\Pi_h y_h) = \int_\Omega \psi_h^*(\cdot, \operatorname{div} y_h)\,\mathrm{d}x .$$

→ **Integral representation**: For every $y_h \in \mathcal{RT}_N^0(\mathcal{T}_h)$, we have that

$$D_h^0(\Pi_h y_h) = -\int_\Omega \phi_h^*(\cdot, \Pi_h y_h)\,\mathrm{d}x - \int_\Omega \psi_h^*(\cdot, \operatorname{div} y_h)\,\mathrm{d}x .$$

- **Assumption**: A maximizer $\Pi_h z_h^{rt} \in \Pi_h(\mathcal{RT}_N^0(\mathcal{T}_h))$, a so-called *discrete dual solution*, exists.

Integral representation of dual problem

- **Integral representation of $F_h^* \circ (-\Lambda_h^*)$:** For every $y_h \in \mathcal{R}T_N^0(\mathcal{T}_h)$, we have that

$$F_h^*(-\Lambda_h^* \Pi_h y_h) = \sup_{v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)} \Big\{ \int_\Omega -\Pi_h y_h \cdot \nabla_h v_h \,\mathrm{d}x - \int_\Omega \psi_h(\cdot, \Pi_h v_h) \,\mathrm{d}x \Big\}$$

$$= \sup_{v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)} \Big\{ \int_\Omega \operatorname{div} y_h \, \Pi_h v_h \,\mathrm{d}x - \int_\Omega \psi_h(\cdot, \Pi_h v_h) \,\mathrm{d}x \Big\}$$

when? $$\Big(= \int_\Omega \psi_h^*(\cdot, \operatorname{div} y_h) \,\mathrm{d}x \Big) .$$

- **Assumption:** For every $y_h \in \mathcal{R}T_N^0(\mathcal{T}_h)$, we have that

$$F_h^*(-\Lambda_h^* \Pi_h y_h) = \int_\Omega \psi_h^*(\cdot, \operatorname{div} y_h) \,\mathrm{d}x .$$

→ **Integral representation:** For every $y_h \in \mathcal{R}T_N^0(\mathcal{T}_h)$, we have that

$$D_h^{rt}(y_h) \coloneqq -\int_\Omega \phi_h^*(\cdot, \Pi_h y_h) \,\mathrm{d}x - \int_\Omega \psi_h^*(\cdot, \operatorname{div} y_h) \,\mathrm{d}x .$$

- **Assumption:** A maximizer $z_h^{rt} \in \mathcal{R}T_N^0(\mathcal{T}_h)$, a so-called *discrete dual solution*, exists.

Lemma: (discrete weak duality relation)

There holds a *discrete weak duality relation*, *i.e.*, it holds that

$$\inf_{v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)} I_h^{cr}(v_h) \geq \sup_{y_h \in \mathcal{RT}^0_N(\mathcal{T}_h)} D_h^{rt}(y_h)\,.$$

♦ **Proof (for integral functionals).** Let $v_h \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ and $y_h \in \mathcal{RT}^0_N(\mathcal{T}_h)$ be arbitrary.

- By the *Fenchel–Young inequality*, it holds that

$$\left.\begin{aligned} \Pi_h y_h \cdot \nabla_h v_h &\leq \phi_h^*(\cdot, \Pi_h y_h) + \phi_h(\cdot, \nabla_h v_h) && \text{a.e. in } \Omega\,,\\ \operatorname{div} y_h\, \Pi_h v_h &\leq \psi_h^*(\cdot, \operatorname{div} y_h) + \psi_h(\cdot, \Pi_h v_h) && \text{a.e. in } \Omega\,. \end{aligned}\right\} \quad (*)$$

- Summation of $(*)$ and the *discrete integration-by-parts formula* yield that

$$\begin{aligned} 0 &= \int_\Omega \Pi_h y_h \cdot \nabla_h v_h \,\mathrm{d}x + \int_\Omega \operatorname{div} y_h\, \Pi_h v_h \,\mathrm{d}x \\ &\leq \int_\Omega \phi_h(\cdot, \nabla_h v_h)\,\mathrm{d}x + \int_\Omega \psi_h(\cdot, \Pi_h v_h)\,\mathrm{d}x \\ &\quad + \int_\Omega \phi_h^*(\cdot, \Pi_h y_h)\,\mathrm{d}x + \int_\Omega \psi_h^*(\cdot, \operatorname{div} y_h)\,\mathrm{d}x \\ &= I_h^{cr}(v_h) - D_h^{rt}(y_h)\,. \end{aligned}$$

■

Lemma: (discrete strong duality ⇔ discrete convex optimality relations)

For $u_h^{cr} \in \mathcal{S}^{1,cr}_D(\mathcal{T}_h)$ and $z_h^{rt} \in \mathcal{R}T^0_N(\mathcal{T}_h)$, the following statements are equivalent:

(i) A *discrete strong duality relation* applies, *i.e.*, it holds that

$$I_h^{cr}(u_h^{cr}) = D_h^{rt}(z_h^{rt})\,;$$

(ii) *Discrete convex optimality relations* apply, *i.e.*, it holds that

$$\begin{aligned}
\phi_h^*(\cdot,\Pi_h z_h^{rt}) - \Pi_h z_h^{rt}\cdot\nabla_h u_h^{cr} + \phi_h(\cdot,\nabla_h u_h^{cr}) &= 0 \qquad \text{a.e. in } \Omega\,,\\
\psi_h^*(\cdot,\operatorname{div} z_h^{rt}) - \operatorname{div} z_h^{rt}\,\Pi_h u_h^{cr} + \psi_h(\cdot,\Pi_h u_h^{cr}) &= 0 \qquad \text{a.e. in } \Omega\,.
\end{aligned}$$

- **Proof.** By the *Fenchel–Young inequality* and *discrete integration-by-parts formula*, it holds that

$$\begin{aligned}
\text{(i)} \quad &\Leftrightarrow\quad 0 = I_h^{cr}(u_h^{cr}) - D_h^{rt}(z_h^{rt})\\
&\Leftrightarrow\quad 0 = \int_\Omega \underbrace{\{\phi_h^*(\cdot,\Pi_h z_h^{rt}) - \Pi_h z_h^{rt}\cdot\nabla_h u_h^{cr} + \phi_h(\cdot,\nabla_h u_h^{cr})\}}_{\geq 0}\,\mathrm{d}x\\
&\qquad\quad + \int_\Omega \underbrace{\{\psi_h^*(\cdot,\operatorname{div} z_h^{rt}) - \operatorname{div} z_h^{rt}\,\Pi_h u_h^{cr} + \psi_h(\cdot,\Pi_h u_h^{cr})\}}_{\geq 0}\,\mathrm{d}x\\
&\Leftrightarrow\quad \text{(ii)}\,.
\end{aligned}$$

■

Discrete reconstruction formula

Lemma: (discrete reconstruction formula)

Let $u_h^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ be a discrete primal solution and let the following be satisfied:

- $\phi_h(x,\cdot) \in C^1(\mathbb{R}^d)$ for a.e. $x \in \Omega$;
- $\psi_h(x,\cdot) \in C^1(\mathbb{R})$ for a.e. $x \in \Omega$;

Then, a dual solution $z_h^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_h)$ is given via

$$z_h^{rt} = D_t\phi_h(\cdot, \nabla_h u_h^{cr}) + \frac{D_t\psi_h(\cdot, \Pi_h u_h^{cr})}{d}(\mathrm{id}_{\mathbb{R}^d} - \Pi_h \mathrm{id}_{\mathbb{R}^d}) \quad \text{a.e. in } \Omega\,.$$

In particular, a discrete strong duality applies, *i.e.*, $I_h^{cr}(u_h^{cr}) = D_h^{rt}(z_h^{rt})$.

Proof.

- There exists $\widehat{z}_h \in \mathcal{RT}_N^0(\mathcal{T}_h)$ s.t.

$$\operatorname{div} \widehat{z}_h = D_t\psi_h(\cdot, \Pi_h u_h^{cr}) \quad \text{a.e. in } \Omega\,.$$

- For every $v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, it holds that

$$\int_\Omega (z_h^{rt} - \widehat{z}_h)\cdot \nabla_h v_h \,\mathrm{d}x = \int_\Omega \underbrace{\Pi_h z_h^{rt}}_{= D_t\phi_h(\cdot, \nabla_h u_h^{cr})} \cdot \nabla_h v_h \,\mathrm{d}x + \int_\Omega \underbrace{\operatorname{div} \widehat{z}_h}_{= D_t\psi_h(\cdot, \Pi_h u_h^{cr})} \Pi_h v_h \,\mathrm{d}x = 0\,,$$

i.e., $z_h^{rt} - \widehat{z}_h \in (\nabla_h(\mathcal{S}_D^{1,cr}(\mathcal{T}_h)))^\perp$.

Discrete reconstruction formula

- Due to the *discrete orthogonality relation*, it follows that

$$z_h^{rt} - \widehat{z}_h \in (\nabla_h(\mathcal{S}_D^{1,cr}(\mathcal{T}_h)))^{\perp} = \ker(\operatorname{div}|_{\mathcal{RT}_N^0(\mathcal{T}_h)})\,,$$

i.e., we have that $z_h^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_h)$ with

$$\left.\begin{aligned}\operatorname{div} z_h^{rt} &= \operatorname{div}\widehat{z}_h\\ &= D_t\psi_h(\cdot,\Pi_h u_h^{cr})\end{aligned}\right\}\quad \text{a.e. in } \Omega\,.$$

- In summary, we have that $u_h^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ and $z_h^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_h)$ satisfy

$$\begin{cases}\Pi_h z_h^{rt} = D_t\phi_h(\cdot,\nabla_h^{cr} u_h^{cr}) & \text{a.e. in } \Omega\,,\\ \operatorname{div} z_h^{rt} = D_t\psi_h(\cdot,\Pi_h u_h^{cr}) & \text{a.e. in } \Omega\,.\end{cases}$$

$$\Leftrightarrow\quad \begin{cases}\phi_h^*(\cdot,\Pi_h z_h^{rt}) - \Pi_h z_h^{rt}\cdot\nabla_h u_h^{cr} + \phi_h(\cdot,\nabla_h u_h^{cr}) = 0 & \text{a.e. in } \Omega\,,\\ \psi_h^*(\cdot,\operatorname{div} z_h^{rt}) - \operatorname{div} z_h^{rt}\,\Pi_h u_h^{cr} + \psi_h(\cdot,\Pi_h u_h^{cr}) = 0 & \text{a.e. in } \Omega\,.\end{cases}$$

$$\Leftrightarrow\quad I_h^{cr}(u_h^{cr}) = D_h^{rt}(z_h^{rt})\,.$$

→ By the *discrete weak duality relation*, we conclude that

$$\begin{aligned}D_h^{rt}(z_h^{rt}) &= I_h^{cr}(u_h^{cr})\\ &\geq \sup_{y_h\in\mathcal{RT}_N^0(\mathcal{T}_h)} D_h^{rt}(y_h)\,.\end{aligned}$$

■

Discrete reconstruction formula

Lemma: (discrete reconstruction formula)

Let $z_h^{rt} \in \mathcal{R}T_N^0(\mathcal{T}_h)$ be a discrete dual solution and let the following be satisfied:

- $\phi_h^*(x,\cdot) \in C^1(\mathbb{R}^d)$ for a.e. $x \in \Omega$;
- $\psi_h^*(x,\cdot) \in C^1(\mathbb{R})$ for a.e. $x \in \Omega$;

Then, a discrete primal solution $u_h^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ is given via

$$u_h^{cr} = D_t\psi_h^*(\cdot, \operatorname{div} z_h^{rt}) + D_t\phi_h^*(\cdot, \Pi_h z_h^{rt}) \cdot (\mathrm{id}_{\mathbb{R}^d} - \Pi_h \mathrm{id}_{\mathbb{R}^d}) \quad \text{a.e. in } \Omega\,.$$

In particular, a discrete strong duality applies, *i.e.*, $I_h^{cr}(u_h^{cr}) = D_h^{rt}(z_h^{rt})$.

Proof.

- There exists $\widehat{u}_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ s.t.

$$\nabla_h \widehat{u}_h = D_t\phi_h^*(\cdot, \Pi_h z_h^{rt}) \quad \text{a.e. in } \Omega\,.$$

- For every $y_h \in \mathcal{R}T_N^0(\mathcal{T}_h)$, it holds that

$$\int_\Omega (u_h^{cr} - \widehat{u}_h) \cdot \operatorname{div} y_h \,\mathrm{d}x = \int_\Omega \underbrace{\Pi_h u_h^{cr}}_{= D_t\phi_h^*(\cdot, \Pi_h z_h^{rt})} \operatorname{div} y_h \,\mathrm{d}x + \int_\Omega \underbrace{\nabla_h \widehat{u}_h}_{= D_t\psi_h^*(\cdot, \operatorname{div} z_h^{rt})} \cdot \Pi_h y_h \,\mathrm{d}x = 0\,,$$

i.e., $u_h^{cr} - \widehat{u}_h \in (\operatorname{div}(\mathcal{R}T_N^0(\mathcal{T}_h)))^\perp$.

Discrete reconstruction formula

- Due to the *surjectivity of divergence operator*, it follows that

$$u_h^{cr} - \widehat{u}_h \in (\operatorname{div}(\mathcal{RT}_N^0(\mathcal{T}_h)))^{\perp} = \begin{cases} \{0\} & \text{if } \Gamma_N \neq \partial\Omega\,, \\ \mathbb{R} & \text{if } \Gamma_N = \partial\Omega\,, \end{cases}$$

i.e., we have that $u_h^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ with

$$\left.\begin{aligned} \nabla_h u_h^{cr} &= \nabla_h \widehat{u}_h \\ &= D_t\psi_h^*(\cdot, \Pi_h z_h^{rt}) \end{aligned}\right\} \quad \text{a.e. in } \Omega\,.$$

- In summary, we have that $u_h^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ and $z_h^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_h)$ satisfy

$$\begin{cases} \Pi_h u_h^{cr} = D_t\phi_h^*(\cdot, \operatorname{div} z_h^{rt}) \ \text{ a.e. in } \Omega\,, \\ \nabla_h u_h^{cr} = D_t\psi_h^*(\cdot, \Pi_h z_h^{rt}) \ \text{ a.e. in } \Omega\,. \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \phi_h^*(\cdot, \Pi_h z_h^{rt}) - \Pi_h z_h^{rt} \cdot \nabla_h u_h^{cr} + \phi_h(\cdot, \nabla_h u_h^{cr}) = 0 \ \text{ a.e. in } \Omega\,, \\ \psi_h^*(\cdot, \operatorname{div} z_h^{rt}) - \operatorname{div} z_h^{rt}\, \Pi_h u_h^{cr} + \psi_h(\cdot, \Pi_h u_h^{cr}) = 0 \ \text{ a.e. in } \Omega\,. \end{cases}$$

$$\Leftrightarrow \quad I_h^{cr}(u_h^{cr}) = D_h^{rt}(z_h^{rt})\,.$$

→ By the *discrete weak duality relation*, we conclude that

$$\begin{aligned} I_h^{cr}(u_h^{cr}) &= D_h^{rt}(z_h^{rt}) \\ &\leq \inf_{v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)} I_h^{cr}(v_h)\,. \end{aligned}$$

■

Examples

Examples: Poisson problem

- **Discrete primal problem:** Minimize $I_h^{cr} : \mathcal{S}_D^{1,cr}(\mathcal{T}_h) \to \mathbb{R}$, for every $v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ defined by

$$I_h^{cr}(v_h) \coloneqq \frac{1}{2}\int_\Omega |\nabla_h v_h|^2 \,\mathrm{d}x - \int_\Omega f_h \,\Pi_h v_h \,\mathrm{d}x\,, \qquad (f_h \in \mathbb{P}^0(\mathcal{T}_h))$$

  i.e., $\phi_h := \frac{1}{2}|\cdot|^2 \in C^1(\mathbb{R}^d)$ and $\psi_h(x,\cdot) := (t \mapsto -f_h(x)t) \in C^1(\mathbb{R})$ for a.e. $x \in \Omega$.

- **Application:** (Deflection of membrane)

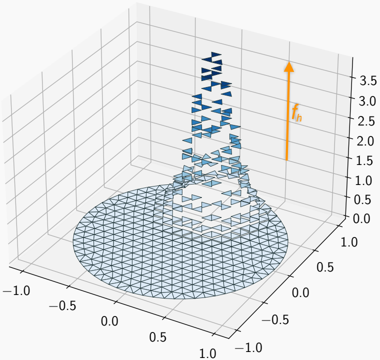


Figure: $f_h \in \mathbb{P}^0(\mathcal{T}_h)$.

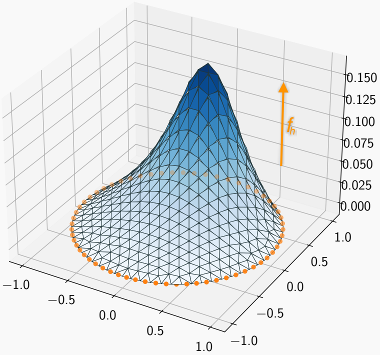


Figure: $u_h^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, where $\Gamma_D = \partial\Omega$.

Examples: Poisson problem

- **Discrete dual problem:** Maximize $D_h^{rt}\colon \mathcal{R}T_N^0(\mathcal{T}_h)\to\mathbb{R}\cup\{-\infty\}$, for every $y_h\in\mathcal{R}T_N^0(\mathcal{T}_h)$ defined by

$$D_h^{rt}(y_h) \coloneqq -\frac{1}{2}\int_\Omega |\Pi_h y_h|^2\,\mathrm{d}x - I_{\{-f_h\}}^{\Omega}(\operatorname{div} y_h)\,,$$

where $I_{\{-f_h\}}^{\Omega}\colon \mathbb{P}^0(\mathcal{T}_h)\to\mathbb{R}\cup\{+\infty\}$, for every $\widehat{v}_h\in\mathbb{P}^0(\mathcal{T}_h)$, is defined by

$$I_{\{-f_h\}}^{\Omega}(\widehat{v}_h) \coloneqq \begin{cases} 0 & \text{if } \widehat{v}_h = -f_h \text{ a.e. in } \Omega\,,\\ +\infty & \text{else}\,.\end{cases}$$

- **Discr. dual solution, discr. strong duality, discr. convex optimality relations:** There exists a discrete dual solution $z_h^{rt}\in\mathcal{R}T_N^0(\mathcal{T}_h)$ s.t.

$$I_h^{cr}(u_h^{cr}) = D_h^{rt}(z_h^{rt}) \quad\Leftrightarrow\quad \begin{cases} \frac{1}{2}|\Pi_h z_h^{rt}|^2 - \Pi_h z_h^{rt}\cdot\nabla_h u_h^{cr} + \frac{1}{2}|\nabla_h u_h|^2 = 0 & \text{a.e. in } \Omega\,,\\ I_{\{-f_h\}}^{(\cdot)}(\operatorname{div} z_h^{rt}) - \operatorname{div} z_h^{rt}\,\Pi_h u_h^{cr} - f_h\,u_h^{cr} = 0 & \text{a.e. in } \Omega\,.\end{cases}$$

$$\Leftrightarrow\quad \begin{cases} \Pi_h z_h^{rt} = \nabla_h u_h^{cr} & \text{a.e. in } \Omega\,,\\ \operatorname{div} z_h^{rt} = -f_h & \text{a.e. in } \Omega\,.\end{cases}$$

- **Discrete primal problem:** Minimize $I_h^{cr}\colon \mathcal{S}_D^{1,cr}(\mathcal{T}_h)\to\mathbb{R}$, for every $v_h\in\mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, defined by

$$I_h^{cr}(v_h) \coloneqq \frac{1}{p}\int_\Omega |\nabla_h v_h|^p\,\mathrm{d}x - \int_\Omega f_h\,\Pi_h v_h\,\mathrm{d}x\,, \qquad (f_h\in\mathbb{P}^0(\mathcal{T}_h))$$

i.e., $\phi_h \coloneqq \frac{1}{p}|\cdot|^p \in C^1(\mathbb{R}^d)$ and $\psi_h(x,\cdot)\coloneqq (t\mapsto -f_h(x)t)\in C^1(\mathbb{R})$ for a.e. $x\in\Omega$.

- **Application:**

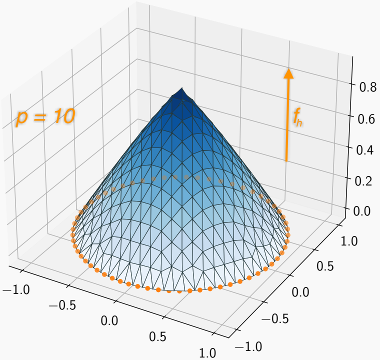


Figure: $u_h^{cr}\in\mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, where $\Gamma_D=\partial B_1^2(0)$ and $f\equiv 1$.

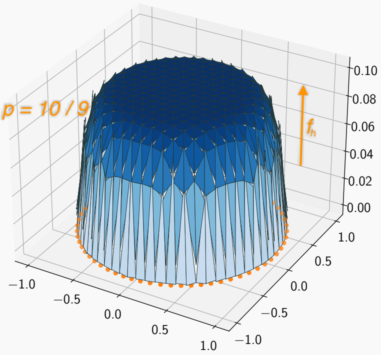


Figure: $u_h^{cr}\in\mathcal{S}_D^{1,cr}(\mathcal{T}_h)$, where $\Gamma_D=\partial B_1^2(0)$ and $f\equiv 1$.

Examples: p-Dirichlet problem

- **Discrete dual problem:** Maximize $D_h^{rt} : \mathcal{RT}_N^0(\mathcal{T}_h) \to \mathbb{R} \cup \{-\infty\}$, for every $y_h \in \mathcal{RT}_N^0(\mathcal{T}_h)$, defined by

$$D_h^{rt}(y_h) \coloneqq -\frac{1}{p'} \int_\Omega |\Pi_h y_h|^{p'} \, \mathrm{d}x - I_{\{-f_h\}}^\Omega(\operatorname{div} y)\,,$$

where $I_{\{-f_h\}}^\Omega : \mathbb{P}^0(\mathcal{T}_h) \to \mathbb{R} \cup \{+\infty\}$, for every $\widehat{v}_h \in \mathbb{P}^0(\mathcal{T}_h)$, is defined by

$$I_{\{-f_h\}}^\Omega(\widehat{v}) \coloneqq \begin{cases} 0 & \text{if } \widehat{v}_h = -f_h \text{ a.e. in } \Omega\,, \\ +\infty & \text{else}\,. \end{cases}$$

- **Discr. dual solution, discr. strong duality, discr. convex optimality relations:** There exists a discrete dual solution $z_h^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_h)$ s.t.

$$I_h^{cr}(u_h^{cr}) = D_h^{rt}(z_h^{rt}) \quad \Leftrightarrow \quad \begin{cases} \frac{1}{p'}|\Pi_h z_h^{rt}|^{p'} - \Pi_h z_h^{rt} \cdot \nabla_h u_h^{cr} + \frac{1}{p}|\nabla_h u_h^{cr}|^p = 0 & \text{a.e. in } \Omega\,, \\ I_{\{-f_h\}}^{(\cdot)}(\operatorname{div} z_h^{rt}) - \operatorname{div} z_h^{rt}\, \Pi_h u_h^{cr} - f_h\, \Pi_h u_h^{cr} = 0 & \text{a.e. in } \Omega\,. \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \Pi_h z_h^{rt} = |\nabla_h u_h^{cr}|^{p-2} \nabla_h u_h^{cr} & \text{a.e. in } \Omega\,, \\ \operatorname{div} z_h^{rt} = -f_h & \text{a.e. in } \Omega\,. \end{cases}$$

Examples: Obstacle problem

- **Discrete primal problem:** Minimize $I_h^{cr} : \mathcal{S}_D^{1,cr}(\mathcal{T}_h) \to \mathbb{R} \cup \{+\infty\}$, for every $v_h \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$ defined by

$$I_h^{cr}(v_h) \coloneqq \frac{1}{2}\int_\Omega |\nabla_h v_h|^2 \,\mathrm{d}x - \int_\Omega f_h \,\Pi_h v_h \,\mathrm{d}x + I_+^\Omega(\Pi_h v_h)\,, \qquad (f_h \in \mathbb{P}^0(\mathcal{T}_h))$$

where $I_+^\Omega : \mathbb{P}^0(\mathcal{T}_h) \to \mathbb{R} \cup \{+\infty\}$, for every $\widehat{v}_h \in \mathbb{P}^0(\mathcal{T}_h)$, is defined by

$$I_+^\Omega(\widehat{v}_h) \coloneqq \begin{cases} 0 & \text{if } \widehat{v}_h \geq 0 \text{ a.e. in } \Omega\,, \\ +\infty & \text{else}\,. \end{cases}$$

- **Application:** (Deflection of membrane with obstacle)

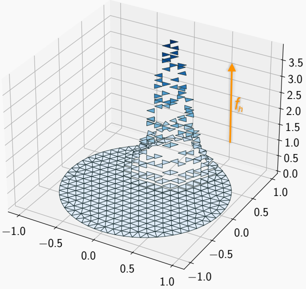


Figure: $f_h \in \mathbb{P}^0(\mathcal{T}_h)$.

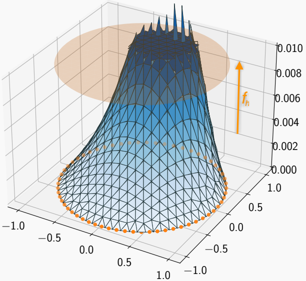


Figure: $u_h^{cr} \in \mathcal{S}_D^{1,cr}(\mathcal{T}_h)$.

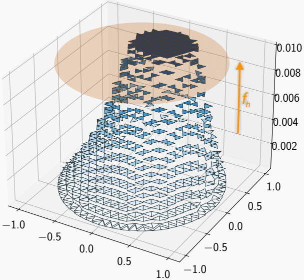


Figure: $\Pi_h u_h^{cr} \in \mathbb{P}^0(\mathcal{T}_h)$.

- **Discrete dual problem:** Maximize $D_h^{rt}: \mathcal{RT}_N^0(\mathcal{T}_h) \to \mathbb{R} \cup \{-\infty\}$, for every $y_h \in \mathcal{RT}_N^0(\mathcal{T}_h)$, defined by

$$D_h^{rt}(y_h) \coloneqq -\frac{1}{2}\int_\Omega |\Pi_h y_h|^2 \, \mathrm{d}x - I_-^\Omega(f_h + \operatorname{div} y_h)\,,$$

where $I_-^\Omega : \mathbb{P}^0(\mathcal{T}_h) \to \mathbb{R} \cup \{+\infty\}$, for every $\widehat{v}_h \in \mathbb{P}^0(\mathcal{T}_h)$, is defined by

$$I_-^\Omega(\widehat{v}_h) \coloneqq \begin{cases} 0 & \text{if } \widehat{v}_h \leq 0 \text{ a.e. in } \Omega\,, \\ +\infty & \text{else}\,. \end{cases}$$

- **Discr. dual solution, discr. strong duality, discr. convex optimality relations:** There exists a discrete dual solution $z_h^{rt} \in \mathcal{RT}_N^0(\mathcal{T}_h)$ s.t.

$$I_h^{cr}(u_h^{cr}) = D_h^{rt}(z_h^{rt}) \quad \Leftrightarrow \quad \begin{cases} \frac{1}{2}|\Pi_h z_h^{rt}|^2 - \Pi_h z_h^{rt} \cdot \nabla_h u_h^{cr} + \frac{1}{2}|\nabla_h u_h^{cr}|^2 = 0 & \text{a.e. in } \Omega\,, \\ \left.\begin{array}{l} I_-^{(\cdot)}(f_h + \operatorname{div} z_h^{rt}) \\ \quad - (f_h + \operatorname{div} z_h^{rt})\,\Pi_h u_h^{cr} + I_-^{(\cdot)}(\Pi_h u_h^{cr}) \end{array}\right\} = 0 & \text{a.e. in } \Omega\,. \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \Pi_h z_h^{rt} = \nabla_h u_h^{cr} & \text{a.e. in } \Omega\,, \\ f_h + \operatorname{div} z_h^{rt} \leq 0 & \text{a.e. in } \Omega\,, \\ \Pi_h u_h^{cr} \geq 0 & \text{a.e. in } \Omega\,, \\ (f_h + \operatorname{div} z_h^{rt})\,\Pi_h u_h^{cr} = 0 & \text{a.e. in } \Omega\,. \end{cases}$$

Thank You for today!

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