A priori and *a posteriori* error identities for convex minimization problems based on convex duality relations

Lecture 2

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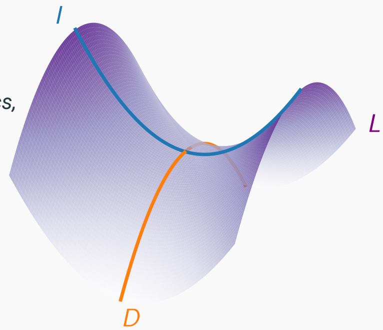


Content of the Lecture 2

- **Lecture 2: Convex duality theory for integral functionals**
  - Fenchel duality theory for integral functionals
    - Integral representation of dual energy functional;
    - Fenchel duality relations;
    - Reconstruction formulas;
    - Examples.
  - *A posteriori* error identity on the basis of convex duality;
    - (Optimal) strong convex measures;
    - Examples;
    - Primal-dual gap identity;
    - Examples.

Fenchel duality theory
for integral functionals

Primal problem

- **Assumptions:**

  (C.1) $\phi\colon \Omega\times\mathbb{R}^d\to\mathbb{R}\cup\{+\infty\}$ measurable s.t. $\phi(x,\cdot)\in\Gamma_0(\mathbb{R}^d)$ for a.e. $x\in\Omega$;

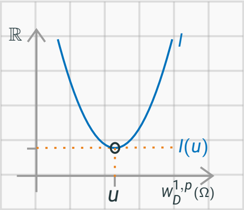
  (C.2) $\psi\colon \Omega\times\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ measurable s.t. $\psi(x,\cdot)\in\Gamma_0(\mathbb{R})$ for a.e. $x\in\Omega$;

  (C.3) For all $y\in(L^p(\Omega))^d$ and $v\in L^p(\Omega)$, it holds that
$$\exists \int_\Omega \phi(\cdot,y)\,\mathrm{d}x,\ \int_\Omega \psi(\cdot,v)\,\mathrm{d}x \in \mathbb{R}\cup\{+\infty\}\,;$$

  (C.4) There exists $v_0\in W^{1,p}_D(\Omega)$ s.t.
$$\exists \int_\Omega \phi(\cdot,\nabla v_0)\,\mathrm{d}x,\ \int_\Omega \psi(\cdot,v_0)\,\mathrm{d}x \in \mathbb{R}\,.$$

- **Primal problem:** Minimize $I\colon W^{1,p}_D(\Omega)\to\mathbb{R}\cup\{+\infty\}$, for every $v\in W^{1,p}_D(\Omega)$ defined by
$$I(v) \coloneqq \int_\Omega \phi(\cdot,\nabla v)\,\mathrm{d}x + \int_\Omega \psi(\cdot,v)\,\mathrm{d}x\,.$$

- **Assumption:** A minimizer $u\in W^{1,p}_D(\Omega)$, a so-called primal solution, exists.



(Fenchel) primal problem

- **Setup of a (Fenchel) primal problem:**
  - Let $G\colon (L^p(\Omega))^d \to \mathbb{R} \cup \{+\infty\}$, for every $y \in (L^p(\Omega))^d$, be defined by

$$G(y) \coloneqq \int_\Omega \phi(\cdot, y)\,\mathrm{d}x\,.$$

→ $G \in \Gamma_0((L^p(\Omega))^d)$ (*cf.* (C.1) & (C.3) & (C.4));

  - Let $F\colon W_D^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$, for every $v \in W_D^{1,p}(\Omega)$, be defined by

$$F(v) \coloneqq \int_\Omega \psi(\cdot, v)\,\mathrm{d}x\,.$$

→ $F \in \Gamma_0(W_D^{1,p}(\Omega))$ (*cf.* (C.2) & (C.3) & (C.4));

  - Let $\Lambda\colon W_D^{1,p}(\Omega) \to (L^p(\Omega))^d$, for every $v \in W_D^{1,p}(\Omega)$, be defined by

$$\Lambda v \coloneqq \nabla v \quad \text{in } (L^p(\Omega))^d\,.$$

→ $\Lambda \in L(W_D^{1,p}(\Omega); (L^p(\Omega))^d)$.

⇒ **(Fenchel) primal problem:** For every $v \in W_D^{1,p}(\Omega)$, we have that

$$I(v) = G(\Lambda v) + F(v)\,.$$

(Fenchel) dual problem

- **Dual problem:** Maximize $D\colon (L^{p'}(\Omega))^d \to \mathbb{R} \cup \{-\infty\}$, for every $y \in (L^{p'}(\Omega))^d$, defined by

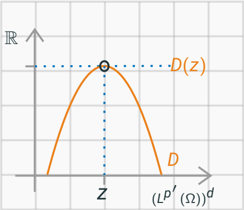
$$D(y) \coloneqq -F^*(-\Lambda^* y) - G^*(y)\,,$$

where

  - For every $y \in (L^{p'}(\Omega))^d$, we have that

$$G^*(y) = \sup_{\widehat{y} \in (L^p(\Omega))^d} \left\{ \int_\Omega y \cdot \widehat{y}\,\mathrm{d}x - \int_\Omega \phi(\cdot, \widehat{y})\,\mathrm{d}x \right\}.$$

when? $$\left(= \int_\Omega \phi^*(\cdot, y)\,\mathrm{d}x\,, \quad \text{where } \phi^*(x,\cdot) := (\phi(x,\cdot))^* \text{ for a.e. } x \in \Omega \right);$$



Lemma: (Fenchel conjugates of integral functionals, *cf.* [1, Rockafellar, '76])

For $\ell \in \mathbb{N}$, let $\Phi\colon \Omega \times \mathbb{R}^\ell \to \mathbb{R} \cup \{+\infty\}$ be a *convex normal integrand*, *i.e.*,

- $\Phi\colon \Omega \times \mathbb{R}^\ell \to \mathbb{R} \cup \{+\infty\}$ is measurable;
- $\Phi(x,\cdot) \in \Gamma_0(\mathbb{R}^\ell)$ for a.e. $x \in \Omega$.

Then, the following statements apply:

(i) $\Phi^*\colon \Omega \times \mathbb{R}^\ell \to \mathbb{R} \cup \{+\infty\}$ is a convex normal integrand;

(ii) If, in addition, for every $y \in (L^p(\Omega))^\ell$, it holds that

$$\exists \int_\Omega \Phi(\cdot,y)\,\mathrm{d}x \in \mathbb{R} \cup \{+\infty\}\,.$$

and $F_\Phi\colon (L^p(\Omega))^\ell \to \mathbb{R} \cup \{+\infty\}$, for every $y \in (L^p(\Omega))^\ell$, is defined by

$$F_\Phi(y) \coloneqq \int_\Omega \Phi(\cdot,y)\,\mathrm{d}x\,,$$

then $(F_\Phi)^*\colon (L^{p'}(\Omega))^\ell \to \mathbb{R} \cup \{+\infty\}$, for every $y \in (L^{p'}(\Omega))^\ell$, is given via

$$(F_\Phi)^*(y) = \begin{cases} \int_\Omega \Phi^*(\cdot,y)\,\mathrm{d}x & \text{if } (\Phi^*(\cdot,y))^- \in L^1(\Omega)\,,\\ +\infty & \text{else}\,. \end{cases}$$

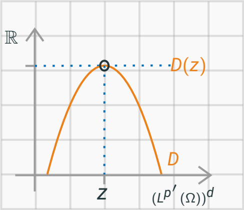
(Fenchel) dual problem

- **Assumption:** (C.5) For all $y \in (L^{p'}(\Omega))^d$ and $v \in L^{p'}(\Omega)$, it holds that

$$\exists \int_\Omega \phi^*(\cdot, y)\,\mathrm{d}x, \int_\Omega \psi^*(\cdot, v)\,\mathrm{d}x \in \mathbb{R} \cup \{+\infty\}\,.$$

- **Dual problem:** Maximize $D\colon (L^{p'}(\Omega))^d \to \mathbb{R} \cup \{-\infty\}$, for every $y \in (L^{p'}(\Omega))^d$, defined by

$$D(y) \coloneqq -F^*(-\Lambda^* y) - G^*(y)\,,$$



where

- For every $y \in (L^{p'}(\Omega))^d$, we have that

$$\begin{aligned} G^*(y) &= \sup_{\widehat{y} \in (L^p(\Omega))^d} \left\{ \int_\Omega y \cdot \widehat{y}\,\mathrm{d}x - \int_\Omega \phi(\cdot, \widehat{y})\,\mathrm{d}x \right\}. \\ &= \int_\Omega \phi^*(\cdot, y)\,\mathrm{d}x\,; \end{aligned}$$

- For every $y \in (L^{p'}(\Omega))^d$, we have that

$$F^*(-\Lambda^* y) = \sup_{v \in W^{1,p}_D(\Omega)} \left\{ \int_\Omega -y \cdot \nabla v\,\mathrm{d}x - \int_\Omega \psi(\cdot, v)\,\mathrm{d}x \right\}.$$

Integral representation of dual problem

- **Integral representation of $F^* \circ (-\Lambda^*)$:** For every $y \in W_N^{p'}(\mathrm{div};\Omega)$, we have that

$$F^*(-\Lambda^* y) = \sup_{v \in W_D^{1,p}(\Omega)} \Big\{ \int_\Omega -y \cdot \nabla v \,\mathrm{d}x - \int_\Omega \psi(\cdot, v)\,\mathrm{d}x \Big\}$$

$$= \sup_{v \in W_D^{1,p}(\Omega)} \Big\{ \int_\Omega \mathrm{div}\, y\, v \,\mathrm{d}x - \int_\Omega \psi(\cdot, v)\,\mathrm{d}x \Big\}$$

when? $$\Big(= \int_\Omega \psi^*(\cdot, \mathrm{div}\, y)\,\mathrm{d}x\,, \quad \text{where } \psi^*(x,\cdot) := (\psi(x,\cdot))^* \text{ for a.e. } x \in \Omega \Big)\,.$$

Fenchel conjugate of integral functionals defined on $W^{1,p}_D(\Omega)$

Lemma: (Fenchel conjugate of integral functionals defined on $W^{1,p}_D(\Omega)$)

Let the following assumptions be satisfied:

(i) $\psi\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a *Carathéodory mapping*, *i.e.*,
- $\psi(\cdot,t)\colon\Omega\to\mathbb{R}$ is measurable for all $t\in\mathbb{R}$;
- $\psi(x,\cdot)\colon\mathbb{R}\to\mathbb{R}$ is continuous for a.e. $x\in\Omega$.

(ii) For every $v\in L^p(\Omega)$, it holds that $\psi(\cdot,v)\in L^1(\Omega)$.

Then, for every $\hat{v}\in L^{p'}(\Omega)$, it holds that

$$\int_\Omega \psi^*(\cdot,\hat{v})\,\mathrm{d}x = \sup_{v\in W^{1,p}_D(\Omega)}\left\{\int_\Omega \hat{v}\,v\,\mathrm{d}x - \int_\Omega \psi(\cdot,v)\,\mathrm{d}x\right\}.$$

Proof.

- From (C.2)–(C.4) and (i), (ii), it follows that

$$\overline{F} \coloneqq \left(v\mapsto \int_\Omega \psi(\cdot,v)\,\mathrm{d}x\right)\in C^0(L^p(\Omega)).$$

→ For every $\hat{v}\in L^{p'}(\Omega)$, we find that

$$\sup_{v\in W^{1,p}_D(\Omega)}\left\{\int_\Omega \hat{v}\,v\,\mathrm{d}x - \int_\Omega \psi(\cdot,v)\,\mathrm{d}x\right\} = \sup_{v\in L^p(\Omega)}\left\{\int_\Omega \hat{v}\,v\,\mathrm{d}x - \int_\Omega \psi(\cdot,v)\,\mathrm{d}x\right\}. \quad ■$$

Integral representation of dual problem

- **Integral representation of $F^* \circ (-\Lambda^*)$:** For every $y \in W_N^{p'}(\mathrm{div};\Omega)$, we have that

$$
\begin{aligned}
F^*(-\Lambda^* y) &= \sup_{v \in W_D^{1,p}(\Omega)} \Big\{ \int_\Omega -y \cdot \nabla v \,\mathrm{d}x - \int_\Omega \psi(\cdot, v)\,\mathrm{d}x \Big\} \\
&= \sup_{v \in W_D^{1,p}(\Omega)} \Big\{ \int_\Omega \mathrm{div}\, y\, v \,\mathrm{d}x - \int_\Omega \psi(\cdot, v)\,\mathrm{d}x \Big\} \\
\text{when?} \qquad &\Big(= \int_\Omega \psi^*(\cdot, \mathrm{div}\, y)\,\mathrm{d}x\,, \quad \text{where } \psi^*(x,\cdot) := (\psi(x,\cdot))^* \text{ for a.e. } x \in \Omega \Big)\,.
\end{aligned}
$$

- **Assumption:** (C.6) For every $y \in W_N^{p'}(\mathrm{div};\Omega)$, we have that

$$
F^*(-\Lambda^* y) = \int_\Omega \psi^*(\cdot, \mathrm{div}\, y)\,\mathrm{d}x\,.
$$

$\Rightarrow$ **Integral representation:** For every $y \in W_N^{p'}(\mathrm{div};\Omega)$, we have that

$$
D(y) = -\int_\Omega \phi^*(\cdot, y)\,\mathrm{d}x - \int_\Omega \psi^*(\cdot, \mathrm{div}\, y)\,\mathrm{d}x\,.
$$

Weak duality relation

Lemma: (weak duality relation)

There holds a *weak duality relation*, *i.e.*, it holds that

$$\inf_{v \in W^{1,p}_D(\Omega)} I(v) \geq \sup_{y \in W^{p'}_N(\mathrm{div};\Omega)} D(y)\,.$$

- **Proof (for integral representation).** Let $v \in W^{1,p}_D(\Omega)$ and $y \in W^{p'}_N(\mathrm{div};\Omega)$ be arbitrary.
  - By the *Fenchel–Young inequality*, it holds that

$$\left.\begin{aligned} y \cdot \nabla v &\leq \phi^*(\cdot, y) + \phi(\cdot, \nabla v) &&\text{a.e. in } \Omega\,,\\ \mathrm{div}\, y\, v &\leq \psi^*(\cdot, \mathrm{div}\, y) + \psi(\cdot, v) &&\text{a.e. in } \Omega\,. \end{aligned}\right\} \quad (*)$$

  - Summation of $(*)$ and the *integration-by-parts formula* yield that

$$\begin{aligned} 0 &= \int_\Omega y \cdot \nabla v \,\mathrm{d}x + \int_\Omega \mathrm{div}\, y\, v \,\mathrm{d}x \\ &\leq \int_\Omega \phi(\cdot, \nabla v) \,\mathrm{d}x + \int_\Omega \psi(\cdot, v) \,\mathrm{d}x \\ &\quad + \int_\Omega \phi^*(\cdot, y) \,\mathrm{d}x + \int_\Omega \psi^*(\cdot, \mathrm{div}\, y) \,\mathrm{d}x \\ &= I(v) - D(y)\,. \end{aligned}$$

■

Lemma: (strong duality relation ⇔ convex optimality relations)

For $u \in W^{1,p}_D(\Omega)$ and $z \in W^{p'}_N(\mathrm{div};\Omega)$, the following statements are equivalent:

(i) A *strong duality relation* applies, *i.e.*, it holds that

$$I(u) = D(z)\,;$$

(ii) *Convex optimality relations* apply, *i.e.*, it holds that

$$\begin{aligned} \phi^*(\cdot,z) - z\cdot\nabla u + \phi(\cdot,\nabla u) &= 0 \qquad \text{a.e. in } \Omega\,,\\ \psi^*(\cdot,\operatorname{div} z) - \operatorname{div} z\, u + \psi(\cdot,u) &= 0 \qquad \text{a.e. in } \Omega\,. \end{aligned}$$

- **Proof.** By the *Fenchel–Young inequality* and the *integration-by-parts formula*, it holds that

$$\begin{aligned} \text{(i)} \quad &\Leftrightarrow \quad 0 = I(u) - D(z)\\ &\Leftrightarrow \quad 0 = \int_\Omega \underbrace{\{\phi^*(\cdot,z) - z\cdot\nabla u + \phi(\cdot,\nabla u)\}}_{\geq 0}\,\mathrm{d}x\\ &\qquad\qquad + \int_\Omega \underbrace{\{\psi^*(\cdot,\operatorname{div} z) - \operatorname{div} z\, u + \psi(\cdot,u)\}}_{\geq 0}\,\mathrm{d}x\\ &\Leftrightarrow \quad \text{(ii)}\,. \end{aligned}$$

■

Sufficient conditions for strong duality

Lemma: (sufficient conditions for strong duality)

Let the following assumptions be satisfied:

(i) $\phi\colon \Omega\times\mathbb{R}^d\to\mathbb{R}$ is a *Carathéodory mapping*, *i.e.*,
- $\phi(\cdot,t)\colon\Omega\to\mathbb{R}$ is measurable for all $t\in\mathbb{R}^d$;
- $\phi(x,\cdot)\colon\mathbb{R}^d\to\mathbb{R}$ is continuous for a.e. $x\in\Omega$.

(ii) For every $y\in(L^p(\Omega))^d$, it holds that $\phi(\cdot,y)\in L^1(\Omega)$.

Then, a strong duality relation applies, *i.e.*, it holds that

$$I(u)=D(z)\,.$$

Proof.

- By (C.1)–(C.4) and (i), (ii), we have that

$$G\coloneqq\Big(y\mapsto\int_\Omega\phi(\cdot,y)\,\mathrm{d}x\Big)\in\Gamma_0((L^p(\Omega))^d)\cap C^0((L^p(\Omega))^d)\,.$$

- By (C.1)–(C.4), we have that

$$F\coloneqq\Big(v\mapsto\int_\Omega\psi(\cdot,v)\,\mathrm{d}x\Big)\in\Gamma_0(W^{1,p}_D(\Omega))\,.$$

→ From the *Fenchel duality theorem*, we conclude that $I(u)=D(z)$. ■

Reconstruction formula

Lemma: (reconstruction formula)

Let $u \in W^{1,p}_D(\Omega)$ is a primal solution and let the following assumptions be satisfied:

- $\phi(x,\cdot) \in C^1(\mathbb{R}^d)$ with $D_t\phi(x,t) \lesssim (1+|t|^{p-1})$ for all $t \in \mathbb{R}^d$ and a.e. $x \in \Omega$;
- $\psi(x,\cdot) \in C^1(\mathbb{R})$ with $D_t\psi(x,t) \lesssim (1+|t|^{p-1})$ for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$.

Then, a dual solution $z \in W^{p'}_N(\operatorname{div};\Omega)$ is given via

$$z = D_t\phi(\cdot,\nabla u) \quad \text{a.e. in } \Omega\,.$$

In particular, a strong duality relation applies, *i.e.*, $I(u) = D(z)$.

Key ingredient I: surjectivity of divergence operator

Lemma: (key ingredient I: surjectivity of divergence operator)

The following statements apply:

(i) If $\Gamma_N \neq \partial\Omega$, then $\operatorname{div}\colon W_N^{p'}(\operatorname{div};\Omega) \to L^{p'}(\Omega)$ is surjective;

(ii) If $\Gamma_N = \partial\Omega$, then $\operatorname{div}\colon W_N^{p'}(\operatorname{div};\Omega) \to L_0^{p'}(\Omega) := L^{p'}(\Omega)/\mathbb{R}$ is surjective.

Proof.

ad (i). For $f \in L^{p'}(\Omega)$, choose $y := -\nabla u \in (L^{p'}(\Omega))^d$, where $u \in W_D^{1,p'}(\Omega)$ solves

$$\left.\begin{aligned} -\Delta u &= f && \text{a.e. in } \Omega\,,\\ \nabla u\cdot n &= 0 && \text{a.e. on } \Gamma_N\,,\\ u &= 0 && \text{a.e. on } \Gamma_D\,. \end{aligned}\right\} \qquad \text{(Zarembda)}$$

Then, $y \in W_N^{p'}(\operatorname{div};\Omega)$ with $\operatorname{div} y = f$ a.e. in Ω.

ad (ii). For $f \in L_0^{p'}(\Omega)$, choose $y := -\nabla u \in (L^{p'}(\Omega))^d$, where $u \in W^{1,p'}(\Omega)$ solves

$$\left.\begin{aligned} -\Delta u &= f && \text{a.e. in } \Omega\,,\\ \nabla u\cdot n &= 0 && \text{a.e. on } \partial\Omega\,. \end{aligned}\right\} \qquad \text{(Neumann)}$$

Then, $y \in W_N^{p'}(\operatorname{div};\Omega)$ with $\operatorname{div} y = f$ a.e. in Ω. ∎

Reconstruction formula

Lemma: (reconstruction formula)

Let $u \in W^{1,p}_D(\Omega)$ is a primal solution and let the following assumptions be satisfied:

- $\phi(x,\cdot) \in C^1(\mathbb{R}^d)$ with $D_t\phi(x,t) \lesssim (1+|t|^{p-1})$ for all $t \in \mathbb{R}^d$ and a.e. $x \in \Omega$;
- $\psi(x,\cdot) \in C^1(\mathbb{R})$ with $D_t\psi(x,t) \lesssim (1+|t|^{p-1})$ for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$.

Then, a dual solution $z \in W^{p'}_N(\mathrm{div};\Omega)$ is given via

$$z = D_t\phi(\cdot,\nabla u) \quad \text{a.e. in } \Omega\,.$$

In particular, a strong duality relation applies, *i.e.*, $I(u) = D(z)$

Proof.

- There exists $\widehat{z} \in W^{p'}_N(\mathrm{div};\Omega)$ s.t.

$$\mathrm{div}\,\widehat{z} = D_t\psi(\cdot,u) \quad \text{a.e. in } \Omega\,.$$

- For every $v \in W^{1,p}_D(\Omega)$, it holds that

$$\int_\Omega (z-\widehat{z})\cdot\nabla v\,\mathrm{d}x = \int_\Omega \underbrace{z}_{=D_t\phi(\cdot,\nabla u)}\cdot\nabla v\,\mathrm{d}x + \int_\Omega \underbrace{\mathrm{div}\,\widehat{z}}_{=D_t\psi(\cdot,u)}\,v\,\mathrm{d}x = 0\,,$$

i.e., $z-\widehat{z} \in (\nabla(W^{1,p}_D(\Omega)))^\perp$.

Key ingredient II: orthogonality relation

Lemma: (key ingredient II: orthogonality relation)

$$W_N^{p'}(\operatorname{div}=0;\Omega) = (\nabla(W_D^{1,p}(\Omega)))^{\perp} \qquad (\text{in } (L^{p'}(\Omega))^d)\,.$$

Proof.

ad '$\subseteq$'. For $y \in W_N^{p'}(\operatorname{div}=0;\Omega)$, for every $v \in W_D^{1,p}(\Omega)$, it holds that

$$\int_\Omega \nabla v \cdot y \,\mathrm{d}x = -\int_\Omega v \underbrace{\operatorname{div} y}_{=0} \,\mathrm{d}x = 0\,,$$

i.e., $y \in (\nabla(W_D^{1,p}(\Omega)))^{\perp}$.

ad '$\supseteq$'. For $y \in (\nabla(W_D^{1,p}(\Omega)))^{\perp}$, for every $v \in W_0^{1,p}(\Omega)$, it holds that

$$\int_\Omega \nabla v \cdot y \,\mathrm{d}x = 0\,,$$

i.e., $y \in W^{p'}(\operatorname{div};\Omega)$ with $\operatorname{div} y = 0$ a.e. in Ω.

→ For every $v \in W_D^{1,p}(\Omega)$, it holds that

$$\langle y\cdot n, v\rangle_{W^{1-\frac{1}{p},p}(\partial\Omega)} = \underbrace{\int_\Omega \nabla v \cdot y \,\mathrm{d}x}_{=0} + \int_\Omega v \underbrace{\operatorname{div} y}_{=0} \,\mathrm{d}x = 0\,,$$

i.e., $y \in W_N^{p'}(\operatorname{div}=0;\Omega)$. ■

Reconstruction formula

- Due to the *orthogonality relation*, it follows that

$$z - \widehat{z} \in (\nabla(W_D^{1,p}(\Omega)))^{\perp} = W_N^{p'}(\mathrm{div}=0;\Omega)\,,$$

i.e., we have that $z \in W_N^{p'}(\mathrm{div};\Omega)$ and

$$\left.\begin{aligned} \mathrm{div}\, z &= \mathrm{div}\, \widehat{z} \\ &= D_t\psi(\cdot, u) \end{aligned}\right\} \quad \text{a.e. in } \Omega\,.$$

→ In summary, we have that $u \in W_D^{1,p}(\Omega)$ and $z \in W_N^{p'}(\mathrm{div};\Omega)$ satisfy

$$\left.\begin{aligned} z &= D_t\phi(\cdot,\nabla u) \;\text{ a.e. in } \Omega\,, \\ \mathrm{div}\, z &= D_t\psi(\cdot,u) \quad\;\text{ a.e. in } \Omega\,. \end{aligned}\right\} \;\Leftrightarrow\; \left\{\begin{aligned} \phi^*(\cdot,z) - z\cdot\nabla u + \phi(\cdot,\nabla u) &= 0 \;\text{ a.e. in } \Omega\,, \\ \psi^*(\cdot,\mathrm{div}\, z) - \mathrm{div}\, z\, u + \psi(\cdot,u) &= 0 \;\text{ a.e. in } \Omega\,. \end{aligned}\right.$$

$$\Leftrightarrow \quad I(u) = D(z)\,.$$

→ By the *weak duality relation*, we conclude that

$$\begin{aligned} D(z) &= I(u) \\ &\geq \sup_{y \in W_N^{p'}(\mathrm{div};\Omega)} D(y)\,. \end{aligned}$$

■

Reconstruction formula

Lemma: (reconstruction formula)

Let $z \in W^{p'}_N(\mathrm{div};\Omega)$ be a dual solution let the following assumptions be satisfied:

- $\phi^*(x,\cdot) \in C^1(\mathbb{R}^d)$ with $D_t\phi^*(x,t) \lesssim (1+|t|^{p'-1})$ for all $t \in \mathbb{R}^d$ and a.e. $x \in \Omega$;
- $\psi^*(x,\cdot) \in C^1(\mathbb{R})$ with $D_t\psi^*(x,t) \lesssim (1+|t|^{p'-1})$ for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$.

Then, a primal solution $u \in W^{1,p}_D(\Omega)$ is given via

$$u = D_t\psi^*(\cdot, \operatorname{div} z) \quad \text{a.e. in } \Omega\,.$$

In particular, a strong duality relation, *i.e.*, $I(u) = D(z)$, applies.

Proof.

- There exists $\widehat{u} \in W^{1,p}_D(\Omega)$ s.t.

$$\nabla \widehat{u} = D_t\phi^*(\cdot, z) \quad \text{a.e. in } \Omega\,.$$

- For every $y \in W^{p'}_N(\mathrm{div};\Omega)$, it holds that

$$\int_\Omega (u - \widehat{u}) \operatorname{div} y \,\mathrm{d}x = \int_\Omega \underbrace{u}_{= D_t\psi^*(\cdot, \operatorname{div} z)} \operatorname{div} y \,\mathrm{d}x + \int_\Omega \underbrace{\nabla \widehat{u}}_{= D_t\phi^*(\cdot, z)} \cdot y \,\mathrm{d}x = 0\,,$$

i.e., $u - \widehat{u} \in (\operatorname{div}(W^{p'}_N(\mathrm{div};\Omega)))^\perp$.

Reconstruction formula

- Due to the *surjectivity of divergence operator*, it follows that

$$u - \widehat{u} \in (\operatorname{div}(W^{p'}_N(\operatorname{div};\Omega)))^{\perp} = \begin{cases} \{0\} & \text{if } \Gamma_N \neq \partial\Omega\,, \\ \mathbb{R} & \text{if } \Gamma_N = \partial\Omega\,, \end{cases}\,,$$

i.e., we have that $u \in W^{1,p}(\Omega)$ and

$$\left.\begin{aligned} \nabla u &= \nabla \widehat{u} \\ &= D_t\phi^*(\cdot, z) \end{aligned}\right\} \quad \text{a.e. in } \Omega\,.$$

→ In summary, we have that $u \in W^{1,p}_D(\Omega)$ and $z \in W^{p'}_N(\operatorname{div};\Omega)$ satisfy

$$\left.\begin{aligned} \nabla u &= D_t\phi^*(\cdot, z) && \text{a.e. in } \Omega\,, \\ u &= D_t\psi(\cdot, \operatorname{div} z) && \text{a.e. in } \Omega\,. \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} \phi^*(\cdot, z) - z\cdot\nabla u + \phi(\cdot, \nabla u) &= 0 \;\text{ a.e. in } \Omega\,, \\ \psi^*(\cdot, \operatorname{div} z) - \operatorname{div} z\, u + \psi(\cdot, u) &= 0 \;\text{ a.e. in } \Omega\,. \end{aligned}\right.$$

$$\Leftrightarrow \quad I(u) = D(z)\,.$$

→ By the *weak duality relation*, we conclude that

$$\begin{aligned} I(u) &= D(z) \\ &\leq \inf_{v \in W^{1,p}_D(\Omega)} I(v)\,. \end{aligned}$$

■

Examples

- **Classical formulation:** For $f\colon \Omega \to \mathbb{R}$, seek $u\colon \Omega \to \mathbb{R}$ s.t.

$$\begin{aligned} -\Delta u &= f && \text{a.e. in } \Omega\,, \\ \nabla u \cdot n &= 0 && \text{a.e. on } \Gamma_N\,, \\ u &= 0 && \text{a.e. on } \Gamma_D\,. \end{aligned}$$

- **Application:** (Deflection of membrane)

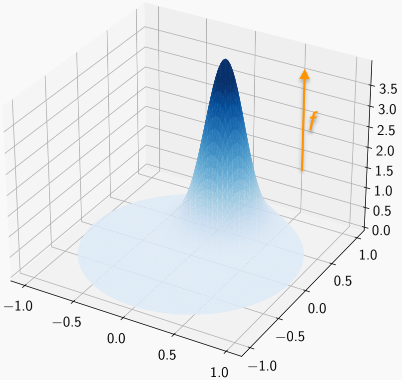


Figure: Load $f \coloneqq 4\exp\left(16|(\cdot) - \frac{6}{10}\mathrm{e}_2|^2\right)\colon B_1^2(0) \to \mathbb{R}$.

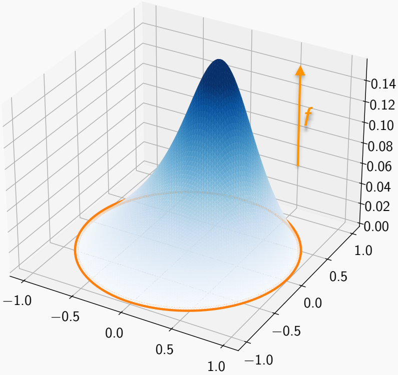


Figure: Membrane $u\colon B_1^2(0) \to \mathbb{R}$, where $\Gamma_D = \partial B_1^2(0)$.

Examples: Poisson problem

- **Primal problem:** Minimize $I\colon W^{1,2}_D(\Omega) \to \mathbb{R}$, for every $v \in W^{1,2}_D(\Omega)$ defined by

$$I(v) \coloneqq \frac{1}{2}\int_\Omega |\nabla v|^2 \,\mathrm{d}x - \int_\Omega f\, v \,\mathrm{d}x\,, \qquad (f \in L^2(\Omega))$$

i.e., $\phi \coloneqq \frac{1}{2}|\cdot|^2 \in C^1(\mathbb{R}^d)$ and $\psi(x,\cdot) \coloneqq (t \mapsto -f(x)t) \in C^1(\mathbb{R})$ for a.e. $x \in \Omega$.

- **Application:** (Deflection of membrane)

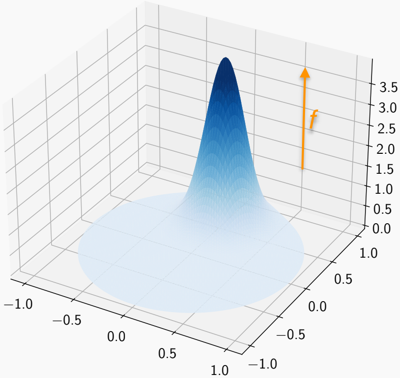


Figure: Load $f \coloneqq 4\exp\big(16|(\cdot) - \frac{6}{10}\mathrm{e}_2|^2\big)\colon B_1^2(0) \to \mathbb{R}$.

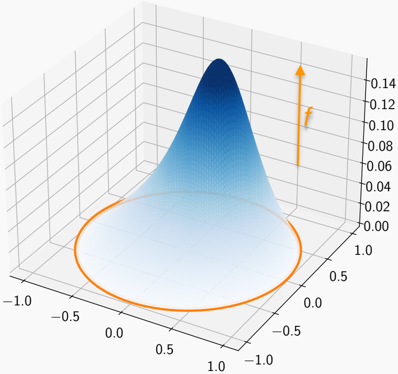


Figure: Membrane $u\colon B_1^2(0) \to \mathbb{R}$, where $\Gamma_D = \partial B_1^2(0)$.

Examples: Poisson problem

- **Dual problem:** Maximize $D\colon W^2_N(\mathrm{div};\Omega)\to\mathbb{R}\cup\{-\infty\}$, for every $y\in W^2_N(\mathrm{div};\Omega)$ defined by

$$D(y) \coloneqq -\frac{1}{2}\int_\Omega |y|^2\,\mathrm{d}x - I^\Omega_{\{-f\}}(\operatorname{div} y)\,,$$

where $I^\Omega_{\{-f\}}\colon L^2(\Omega)\to\mathbb{R}\cup\{+\infty\}$, for every $\widehat{v}\in L^2(\Omega)$, is defined by

$$I^\Omega_{\{-f\}}(\widehat{v}) \coloneqq \begin{cases} 0 & \text{if } \widehat{v}=-f \text{ a.e. in } \Omega\,,\\ +\infty & \text{else}\,. \end{cases}$$

- **Existence of a dual solution, strong duality, and convex optimality relations:** There exists a unique dual solution $z\in W^2_N(\mathrm{div};\Omega)$ s.t.

$$I(u)=D(z) \quad\Leftrightarrow\quad \begin{cases} \frac{1}{2}|z|^2 - z\cdot\nabla u + \frac{1}{2}|\nabla u|^2 = 0 & \text{a.e. in } \Omega\,,\\ I^{(\cdot)}_{\{-f\}}(\operatorname{div} z) - \operatorname{div} z\, u - f\,u = 0 & \text{a.e. in } \Omega\,. \end{cases}$$

$$\Leftrightarrow\quad \begin{cases} z = \nabla u & \text{a.e. in } \Omega\,,\\ \operatorname{div} z = -f & \text{a.e. in } \Omega\,. \end{cases}$$

Examples: p-Dirichlet problem

- **Classical formulation:** Given $f\colon \Omega \to \mathbb{R}$, seek $u\colon \Omega \to \mathbb{R}$ s.t.

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) &= f && \text{a.e. in } \Omega\,,\\ \nabla u \cdot n &= 0 && \text{a.e. on } \Gamma_N\,,\\ u &= 0 && \text{a.e. on } \Gamma_D\,. \end{aligned}$$

- **Application:**

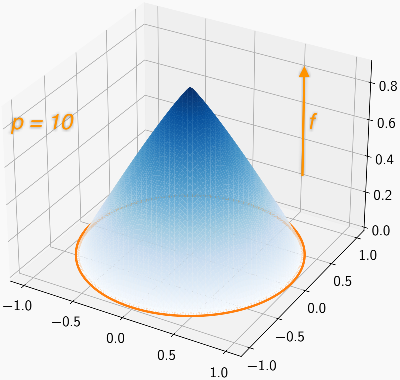


Figure: $u\colon B_1^2(0) \to \mathbb{R}$, where $\Gamma_D = \partial B_1^2(0)$ and $f \equiv 1$.

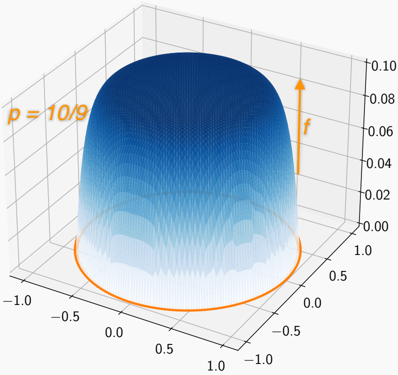


Figure: $u\colon B_1^2(0) \to \mathbb{R}$, where $\Gamma_D = \partial B_1^2(0)$ and $f \equiv 1$.

Examples: p-Dirichlet problem

- **Primal problem:** Minimize $I\colon W_D^{1,p}(\Omega) \to \mathbb{R}$, for every $v \in W_D^{1,p}(\Omega)$ defined by

$$I(v) \coloneqq \frac{1}{p}\int_\Omega |\nabla v|^p \,\mathrm{d}x - \int_\Omega f\, v \,\mathrm{d}x\,, \qquad (f \in L^{p'}(\Omega))$$

i.e., $\phi \coloneqq \frac{1}{p}|\cdot|^p \in C^1(\mathbb{R}^d)$ and $\psi(x,\cdot) \coloneqq (t \mapsto -f(x)t) \in C^1(\mathbb{R})$ for a.e. $x \in \Omega$.

- **Application:**

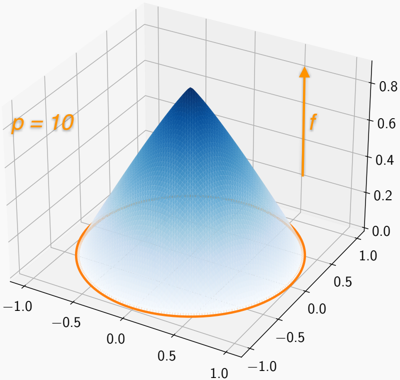


Figure: $u\colon B_1^2(0) \to \mathbb{R}$, where $\Gamma_D = \partial B_1^2(0)$ and $f \equiv 1$.

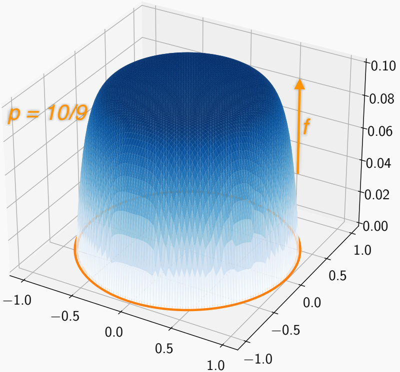


Figure: $u\colon B_1^2(0) \to \mathbb{R}$, where $\Gamma_D = \partial B_1^2(0)$ and $f \equiv 1$.

Examples: p-Dirichlet problem

- **Dual problem:** Maximize $D\colon W_N^{p'}(\operatorname{div};\Omega)\to\mathbb{R}\cup\{-\infty\}$, for every $y\in W_N^2(\operatorname{div};\Omega)$ defined by

$$D(y) \coloneqq -\frac{1}{p'}\int_\Omega |y|^{p'}\,\mathrm{d}x - I_{\{-f\}}^{\Omega}(\operatorname{div} y)\,,$$

  where $I_{\{-f\}}^{\Omega}\colon L^{p'}(\Omega)\to\mathbb{R}\cup\{+\infty\}$, for every $\widehat{v}\in L^{p'}(\Omega)$, is defined by

$$I_{\{-f\}}^{\Omega}(\widehat{v}) \coloneqq \begin{cases} 0 & \text{if } \widehat{v}=-f \text{ a.e. in } \Omega\,,\\ +\infty & \text{else}\,.\end{cases}$$

- **Existence of a dual solution, strong duality, and convex optimality relations:** There exists a unique dual solution $z\in W_N^{p'}(\operatorname{div};\Omega)$ s.t.

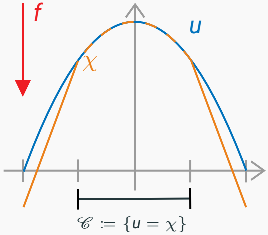
$$I(u) = D(z) \quad\Leftrightarrow\quad \left\{\begin{aligned} \frac{1}{p'}|z|^{p'} - z\cdot\nabla u + \frac{1}{p}|\nabla u|^p &= 0 && \text{a.e. in } \Omega\,,\\ I_{\{-f\}}^{(\cdot)}(\operatorname{div} z) - \operatorname{div} z\, u - f\,u &= 0 && \text{a.e. in } \Omega\,.\end{aligned}\right.$$

$$\Leftrightarrow\quad \left\{\begin{aligned} z &= |\nabla u|^{p-2}\nabla u && \text{a.e. in } \Omega\,,\\ \operatorname{div} z &= -f && \text{a.e. in } \Omega\,.\end{aligned}\right.$$

Examples: Obstacle problem

- **Classical formulation:** For $f\colon \Omega \to \mathbb{R}$ and $\chi\colon \Omega \to \mathbb{R}$, seek $u\colon \Omega \to \mathbb{R}$ s.t.

$$
\begin{aligned}
-\Delta u &\geq f && \text{a.e. in } \Omega\,,\\
u &\geq \chi && \text{a.e. in } \Omega\,,\\
(f+\Delta u)(\chi - u) &= 0 && \text{a.e. in } \Omega\,,\\
\nabla u \cdot n &\geq 0 && \text{a.e. on } \Gamma_N\,,\\
u &= 0 && \text{a.e. on } \Gamma_D\,.
\end{aligned}
$$



- **Application:** (Deflection of membrane with obstacle)

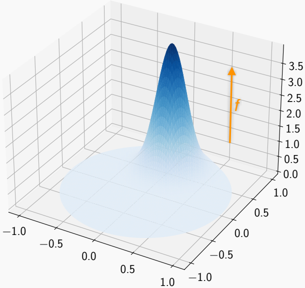


Figure: Load $f\colon B_1^2(0) \to \mathbb{R}$.

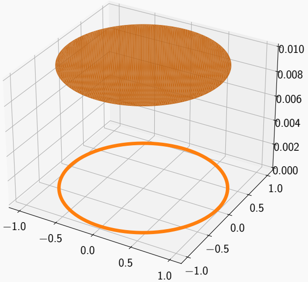


Figure: Obstacle $\chi\colon B_1^2(0) \to \mathbb{R}$.

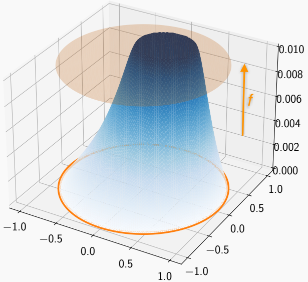


Figure: Membrane $u\colon B_1^2(0) \to \mathbb{R}$.

Examples: Obstacle problem

- **Primal problem:** Minimize $I\colon W^{1,2}_D(\Omega)\to\mathbb{R}\cup\{+\infty\}$, for every $v\in W^{1,2}_D(\Omega)$ defined by

$$I(v) \coloneqq \frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x - \int_\Omega f\,v\,\mathrm{d}x + I_+^\Omega(v)\,, \qquad (f\in L^2(\Omega))$$

where $I_+^\Omega\colon L^2(\Omega)\to\mathbb{R}\cup\{+\infty\}$, for every $\widehat{v}\in L^2(\Omega)$, defined by

$$I_+^\Omega(\widehat{v}) \coloneqq \begin{cases} 0 & \text{if } \widehat{v}\geq 0 \text{ a.e. in } \Omega\,,\\ +\infty & \text{else}\,.\end{cases}$$

- **Application:** (Deflection of membrane with obstacle)

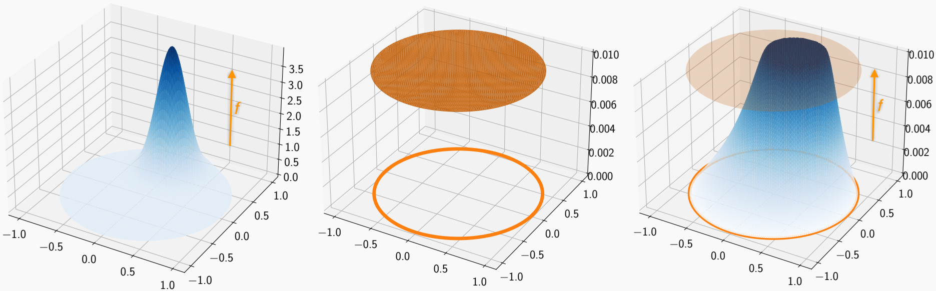


Figure: Load $f\colon B_1^2(0)\to\mathbb{R}$. **Figure:** Obstacle $\chi\colon B_1^2(0)\to\mathbb{R}$. **Figure:** Membrane $u\colon B_1^2(0)\to\mathbb{R}$.

Examples: Obstacle problem

- **Dual problem:** Maximize $D\colon (L^2(\Omega))^d \to \mathbb{R}\cup\{-\infty\}$, for every $y \in (L^2(\Omega))^d$ defined by

$$D(y) \coloneqq -\frac{1}{2}\int_\Omega |y|^2\,\mathrm{d}x - I_-^\Omega(f+\operatorname{div} y)\,,$$

where $I_-^\Omega\colon (W_D^{1,2}(\Omega))^* \to \mathbb{R}\cup\{+\infty\}$, for every $\widehat{v}^* \in (W_D^{1,2}(\Omega))^*$, is defined by

$$I_-^\Omega(\widehat{v}^*) \coloneqq \begin{cases} 0 & \text{if } \widehat{v}^* \leq 0 \text{ in } (W_D^{1,2}(\Omega))^*\,,\\ +\infty & \text{else}\,. \end{cases}$$

- **Existence of a dual solution, strong duality, and convex optimality relations:** There exists a unique dual solution $z \in (L^2(\Omega))^d$ s.t.

$$I(u) = D(z) \quad \Leftrightarrow \quad \begin{cases} \frac{1}{2}|z|^2 - z\cdot\nabla u + \frac{1}{2}|\nabla u|^2 = 0 & \text{a.e. in } \Omega\,,\\ I_-^\Omega(f+\operatorname{div} z) - \langle f+\operatorname{div} z, u\rangle_{W_D^{1,2}(\Omega)} - I_-^\Omega(u) = 0 & \text{a.e. in } \Omega\,. \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} z = \nabla u & \text{a.e. in } \Omega\,,\\ f+\operatorname{div} z \leq 0 & \text{in } (W_D^{1,2}(\Omega))^*\,,\\ u \geq 0 & \text{a.e. in } \Omega\,,\\ \langle f+\operatorname{div} z, u\rangle_{W_D^{1,2}(\Omega)} = 0\,. \end{cases}$$

A posteriori error identity on the basis of convex duality

Strong convexity measure

Definition: (strong convexity measure)

Let X be a Banach space and $x \in X$ minimal for $F\colon X \to \mathbb{R} \cup \{+\infty\}$ with $F(x) < +\infty$.

Then, $\rho_F^2\colon X \times X \to [0, +\infty]$ is called a *strong convexity measure* of F at $x \in X$ if

$$\rho_F^2(y,x) \leq F(y) - F(x) \qquad \text{for all } y \in \operatorname{dom}(F)\,.$$

- **Interpretation:** If $F\colon X \to \mathbb{R} \cup \{+\infty\}$ is twice continuously Fréchet differentiable, then

$$\begin{aligned}
\rho_F^2(y,x) &\leq F(y) - F(x) \\
&= \underbrace{\cancel{F(x) - F(x)}}_{=0} + \langle \underbrace{\cancel{\mathrm{D}F(x)}}_{=0}, y - x\rangle_X \\
&\quad + \int_0^1 \langle \mathrm{D}^2 F(sx + (1-s)y), (x-y)^2\rangle_{X^2}\,\mathrm{d}s \\
&= \underbrace{\int_0^1 \langle \mathrm{D}^2 F(sx + (1-s)y), (x-y)^2\rangle_{X^2}\,\mathrm{d}s}_{\text{quantifies strong convexity}}\,.
\end{aligned}$$

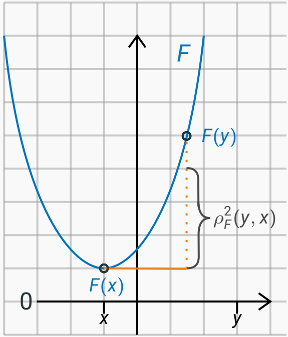


Figure: Interpretation of strong convexity measure in 1D.

Optimal strong convexity measure

Definition: (optimal strong convexity measure)

Let X be a Banach space and $x \in X$ minimal for $F\colon X \to \mathbb{R} \cup \{+\infty\}$ with $F(x) < +\infty$.

Then, the *optimal strong convexity measure* of F at $x \in X$ $\rho^2_{F,\mathrm{opt}}\colon X \times X \to [0,+\infty]$ is defined by

$$\rho^2_{F,\mathrm{opt}}(y,x) \coloneqq F(y) - F(x) \qquad \text{for all } y \in \mathrm{dom}(F)\,.$$

- **Interpretation:** If $F\colon X \to \mathbb{R} \cup \{+\infty\}$ is twice continuously Fréchet differentiable, then

$$\begin{aligned}
\rho^2_{F,\mathrm{opt}}(y,x) &\coloneqq F(y) - F(x) \\
&= \cancel{\underbrace{F(x) - F(x)}_{=0}} + \langle \cancel{\underbrace{\mathrm{D}F(x)}_{=0}}, y - x\rangle_X \\
&\quad + \int_0^1 \langle \mathrm{D}^2 F(sx + (1-s)y), (x-y)^2\rangle_{X^2}\,\mathrm{d}s \\
&= \underbrace{\int_0^1 \langle \mathrm{D}^2 F(sx + (1-s)y), (x-y)^2\rangle_{X^2}\,\mathrm{d}s}_{\text{quantifies strong convexity}}\,.
\end{aligned}$$

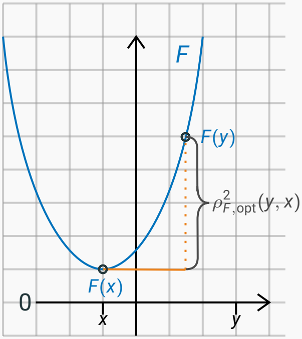


Figure: Interpretation of optimal strong convexity measure in 1D.

Examples

Examples: Poisson problem

- **Primal problem:** Minimize $I\colon W^{1,2}_D(\Omega)\to\mathbb{R}$, for every $v\in W^{1,2}_D(\Omega)$ defined by

$$I(v) \coloneqq \frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x - \int_\Omega f\,v\,\mathrm{d}x\,. \qquad (f\in L^2(\Omega))$$

- **Optimal strong convexity measure:** For every $v\in W^{1,2}_D(\Omega)$, it holds that

$$\begin{aligned}
\rho^2_{I,\mathrm{opt}}(v,u) &= \Big[\frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x - \int_\Omega f\,v\,\mathrm{d}x\Big] - \Big[\frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x - \int_\Omega f\,u\,\mathrm{d}x\Big]\\
&= \cancel{\frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x} + \int_\Omega \underbrace{\nabla u}_{=z}\cdot(\nabla v-\nabla u)\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla v-\nabla u|^2\,\mathrm{d}x\\
&\quad - \cancel{\frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x} - \int_\Omega f\,(v-u)\,\mathrm{d}x\\
&= \cancel{-\int_\Omega \underbrace{\operatorname{div} z}_{=-f}\,(v-u)\,\mathrm{d}x} + \frac{1}{2}\int_\Omega |\nabla v-\nabla u|^2\,\mathrm{d}x\\
&\quad \cancel{-\int_\Omega f\,(v-u)\,\mathrm{d}x}\\
&= \frac{1}{2}\int_\Omega |\nabla v-\nabla u|^2\,\mathrm{d}x\,.
\end{aligned}$$

Examples: Poisson problem

- **Dual problem:** Maximize $D\colon W^2_N(\mathrm{div};\Omega)\to\mathbb{R}\cup\{-\infty\}$, for every $y\in W^2_N(\mathrm{div};\Omega)$ defined by

$$D(y) \coloneqq -\frac{1}{2}\int_\Omega |y|^2\,\mathrm{d}x - I^\Omega_{\{-f\}}(\mathrm{div}\,y)\,. \qquad (f\in L^2(\Omega))$$

- **Optimal strong convexity measure:** For every $y\in W^2_N(\mathrm{div}=-f;\Omega)$, it holds that

$$\begin{aligned}
\rho^2_{-D,\mathrm{opt}}(y,z) &= \Big[\frac{1}{2}\int_\Omega |y|^2\,\mathrm{d}x + \cancel{\underbrace{I^\Omega_{\{-f\}}(\mathrm{div}\,y)}_{=0}}\Big] - \Big[\frac{1}{2}\int_\Omega |z|^2\,\mathrm{d}x + \cancel{\underbrace{I^\Omega_{\{-f\}}(\mathrm{div}\,z)}_{=0}}\Big] \\
&= \cancel{\frac{1}{2}\int_\Omega |z|^2\,\mathrm{d}x} + \int_\Omega \underbrace{z}_{=\nabla u}\cdot(y-z)\,\mathrm{d}x + \frac{1}{2}\int_\Omega |y-z|^2\,\mathrm{d}x \\
&\quad \cancel{-\frac{1}{2}\int_\Omega |z|^2\,\mathrm{d}x}\,. \\
&= \cancel{\int_\Omega (\underbrace{\mathrm{div}\,z}_{=-f} - \underbrace{\mathrm{div}\,y}_{=-f})\,u\,\mathrm{d}x} + \frac{1}{2}\int_\Omega |y-z|^2\,\mathrm{d}x \\
&= \frac{1}{2}\int_\Omega |y-z|^2\,\mathrm{d}x\,.
\end{aligned}$$

Examples: p-Dirichlet problem

- **Primal problem:** Minimize $I\colon W^{1,p}_D(\Omega)\to\mathbb{R}$, for every $v\in W^{1,p}_D(\Omega)$ defined by

$$I(v) \coloneqq \frac{1}{p}\int_\Omega |\nabla v|^p\,\mathrm{d}x - \int_\Omega f\,v\,\mathrm{d}x\,. \qquad (f\in L^{p'}(\Omega))$$

- **Optimal strong convexity measure:** For every $v\in W^{1,p}_D(\Omega)$, it holds that

$$\begin{aligned}
\rho^2_{I,\mathrm{opt}}(v,u) &= \Big[\frac{1}{p}\int_\Omega |\nabla v|^p\,\mathrm{d}x - \int_\Omega f\,v\,\mathrm{d}x\Big] - \Big[\frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x - \int_\Omega f\,u\,\mathrm{d}x\Big]\\
&= \cancel{\frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x} + \cancel{\int_\Omega \underbrace{|\nabla u|^{p-2}\nabla u}_{=z}\cdot(\nabla v-\nabla u)\,\mathrm{d}x}\\
&\quad + \int_\Omega\Big(\int_0^1 |\lambda\nabla v+(1-\lambda)\nabla u|^{p-2}\,\mathrm{d}\lambda\Big)|\nabla v-\nabla u|^2\,\mathrm{d}x\\
&\quad - \cancel{\frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x} - \cancel{\int_\Omega f\,(v-u)\,\mathrm{d}x}\\
&= \int_\Omega\Big(\int_0^1 |\lambda\nabla v+(1-\lambda)\nabla u|^{p-2}\,\mathrm{d}\lambda\Big)|\nabla v-\nabla u|^2\,\mathrm{d}x\\
&\sim \int_\Omega |V(\nabla v)-V(\nabla u)|^2\,\mathrm{d}x\,, \qquad \text{(natural distance)}
\end{aligned}$$

where $V\colon\mathbb{R}^d\to\mathbb{R}^d$ is defined by $V(a)\coloneqq|a|^{\frac{p-2}{2}}a$ for all $a\in\mathbb{R}^d$.

- **Dual problem:** Maximize $D\colon W_N^{p'}(\mathrm{div};\Omega)\to\mathbb{R}\cup\{-\infty\}$, for every $y\in W_N^{p'}(\mathrm{div};\Omega)$ defined by

$$D(y) \coloneqq -\frac{1}{p'}\int_\Omega |y|^{p'}\,\mathrm{d}x - I_{\{-f\}}^{\Omega}(\mathrm{div}\,y)\,. \qquad (f\in L^{p'}(\Omega))$$

- **Optimal strong convexity measure:** For every $y\in W_N^{p'}(\mathrm{div}=-f;\Omega)$, it holds that

$$\begin{aligned}
\rho^2_{-D,\mathrm{opt}}(y,z) &= \Big[\frac{1}{p'}\int_\Omega |y|^{p'}\,\mathrm{d}x + \cancel{\underbrace{I_{\{-f\}}^{\Omega}(\mathrm{div}\,y)}_{=0}}\Big] - \Big[\frac{1}{p'}\int_\Omega |z|^{p'}\,\mathrm{d}x + \cancel{\underbrace{I_{\{-f\}}^{\Omega}(\mathrm{div}\,z)}_{=0}}\Big]\\
&= \cancel{\frac{1}{p'}\int_\Omega |z|^{p'}\,\mathrm{d}x} + \int_\Omega \underbrace{|z|^{p'-2}z}_{=\nabla u}\cdot(y-z)\,\mathrm{d}x\\
&\quad + \int_\Omega \Big(\int_0^1 |\lambda y+(1-\lambda)z|^{p'-2}\,\mathrm{d}\lambda\Big)|y-z|^2\,\mathrm{d}x - \cancel{\frac{1}{p'}\int_\Omega |z|^{p'}\,\mathrm{d}x}\\
&= \cancel{\int_\Omega (\underbrace{\mathrm{div}\,z}_{=-f} - \underbrace{\mathrm{div}\,y}_{=-f})\,u\,\mathrm{d}x} + \int_\Omega \Big(\int_0^1 |\lambda y+(1-\lambda)z|^{p'-2}\,\mathrm{d}\lambda\Big)|y-z|^2\,\mathrm{d}x\\
&\sim \int_\Omega |V^*(y)-V^*(z)|^2\,\mathrm{d}x\,. \qquad \text{(conjugate natural distance)}
\end{aligned}$$

where $V^*\colon\mathbb{R}^d\to\mathbb{R}^d$ is defined by $V^*(a)\coloneqq |a|^{\frac{p'-2}{2}}a$ for all $a\in\mathbb{R}^d$.

- **Primal problem:** Minimize $I\colon W^{1,2}_D(\Omega)\to\mathbb{R}\cup\{+\infty\}$, for every $v\in W^{1,2}_D(\Omega)$ defined by

$$I(v) \coloneqq \frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x - \int_\Omega f\,v\,\mathrm{d}x + I_+^\Omega(v)\,. \qquad (f\in L^2(\Omega))$$

- **Optimal strong convexity measure:** For every $v\in W^{1,2}_D(\Omega;\mathbb{R}_{\geq 0})$, it holds that

$$\begin{aligned}
\rho^2_{I,\mathrm{opt}}(v,u) &= \Big[\frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x - \int_\Omega f\,v\,\mathrm{d}x + \cancel{\underbrace{I_+^\Omega(v)}_{=0}}\Big] - \Big[\frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x - \int_\Omega f\,u\,\mathrm{d}x + \cancel{\underbrace{I_+^\Omega(u)}_{=0}}\Big]\\
&= \cancel{\frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x} + \int_\Omega \underbrace{\nabla u}_{=z}\cdot(\nabla v-\nabla u)\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla v-\nabla u|^2\,\mathrm{d}x\\
&\quad - \int_\Omega f\,(v-u)\,\mathrm{d}x - \cancel{\frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x}\\
&= \frac{1}{2}\int_\Omega |\nabla v-\nabla u|^2\,\mathrm{d}x - \langle f+\operatorname{div} z, v-u\rangle_{W^{1,2}_D(\Omega)}\\
&= \frac{1}{2}\int_\Omega |\nabla v-\nabla u|^2\,\mathrm{d}x - \langle f+\operatorname{div} z, v\rangle_{W^{1,2}_D(\Omega)} + \cancel{\underbrace{\langle f+\operatorname{div} z, u\rangle_{W^{1,2}_D(\Omega)}}_{=0}}\\
&= \frac{1}{2}\int_\Omega |\nabla v-\nabla u|^2\,\mathrm{d}x - \langle f+\operatorname{div} z, v\rangle_{W^{1,2}_D(\Omega)}\,.
\end{aligned}$$

Examples: Obstacle problem

- **Dual problem:** Maximize $D\colon (L^2(\Omega))^d \to \mathbb{R}\cup\{-\infty\}$, for every $y\in (L^2(\Omega))^d$ defined by

$$D(y) \coloneqq -\frac{1}{2}\int_\Omega |y|^2\,\mathrm{d}x - I_-^\Omega(f+\operatorname{div} y)\,. \qquad (f\in L^2(\Omega))$$

- **Optimal strong convexity measure:** For every $y \in W_N^2(\operatorname{div}\leq -f;\Omega)$, it holds that

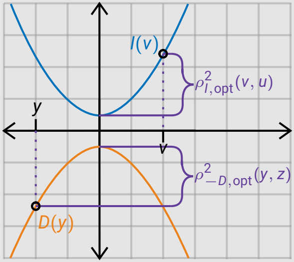
$$\begin{aligned}
\rho^2_{-D,\mathrm{opt}}(y,z) &= \Big[\frac{1}{2}\int_\Omega |y|^2\,\mathrm{d}x + \cancel{\underbrace{I_-^\Omega(\operatorname{div} y + f)}_{=0}}\Big] - \Big[\frac{1}{2}\int_\Omega |z|^2\,\mathrm{d}x + \cancel{\underbrace{I_-^\Omega(\operatorname{div} z + f)}_{=0}}\Big] \\
&= \cancel{\frac{1}{2}\int_\Omega |z|^2\,\mathrm{d}x} + \int_\Omega \underbrace{z}_{=\nabla u}\cdot(y-z)\,\mathrm{d}x + \frac{1}{2}\int_\Omega |y-z|^2\,\mathrm{d}x \\
&\quad - \cancel{\frac{1}{2}\int_\Omega |z|^2\,\mathrm{d}x} \\
&= \frac{1}{2}\int_\Omega |y-z|^2\,\mathrm{d}x + \underbrace{\langle \operatorname{div} z, u\rangle_{W_D^{1,2}(\Omega)}}_{=-\int_\Omega f\,u\,\mathrm{d}x} - \int_\Omega \operatorname{div} y\, u\,\mathrm{d}x \\
&= \frac{1}{2}\int_\Omega |y-z|^2\,\mathrm{d}x - \int_\Omega (f+\operatorname{div} y)\,u\,\mathrm{d}x\,.
\end{aligned}$$

Definition: (primal-dual total error & primal-dual gap estimator)

- The *primal-dual total error* $\rho^2_{\mathrm{tot}}\colon \mathrm{dom}(I)\times\mathrm{dom}(-D)\to[0,+\infty)$ is defined by

$$\rho^2_{\mathrm{tot}}(v,y) \coloneqq \rho^2_{I,\mathrm{opt}}(v,u)+\rho^2_{-D,\mathrm{opt}}(y,z)$$

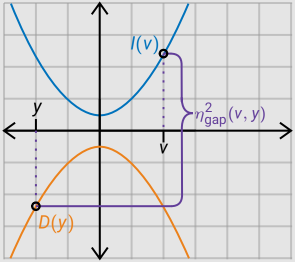
for all $(v,y)^\top\in\mathrm{dom}(I)\times\mathrm{dom}(-D)$.



- The *primal-dual gap estimator* $\eta^2_{\mathrm{gap}}\colon \mathrm{dom}(I)\times\mathrm{dom}(-D)\to[0,+\infty)$ is defined by

$$\eta^2_{\mathrm{gap}}(v,y)\coloneqq I(v)-D(y)$$

for all $(v,y)^\top\in\mathrm{dom}(I)\times\mathrm{dom}(-D)$.



Primal-dual gap identity

Theorem: (primal-dual gap identity)

If a *strong duality relation* applies, *i.e.*,

$$I(u) = D(z)\,,$$

then for every $(v,y)^\top \in \mathrm{dom}(I) \times \mathrm{dom}(-D)$, it holds that

$$\rho^2_{\mathrm{tot}}(v,y) = \eta^2_{\mathrm{gap}}(v,y)\,.$$

- **Proof.** For every $(v,y)^\top \in \mathrm{dom}(I) \times \mathrm{dom}(-D)$, we have that

$$\begin{aligned}
\rho^2_{\mathrm{tot}}(v,u) &\coloneqq \rho^2_{I,\mathrm{opt}}(v,u) + \rho^2_{-D,\mathrm{opt}}(y,z)\\
&\coloneqq [I(v) - I(u)] + [D(z) - D(y)]\\
&= I(v) - \underbrace{[D(z) - I(u)]}_{=0} - D(y)\\
&= I(v) - D(y)\\
&\eqqcolon \eta^2_{\mathrm{gap}}(v,y)\,.
\end{aligned}$$

■

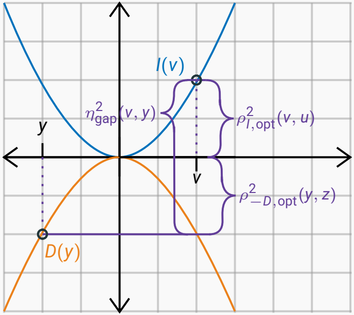


Figure: Primal-dual gap identity in 1D.

Primal-dual gap identity

Theorem: (integral representation of primal-dual gap estimator)

For every $v \in \mathrm{dom}(I)$ and $y \in \mathrm{dom}(-D) \cap W_N^{p'}(\mathrm{div};\Omega)$, it holds that

$$\begin{aligned}\eta_{\mathrm{gap}}^2(v,y) = \int_\Omega \{\phi(\cdot,\nabla v) - \nabla v \cdot y + \phi^*(\cdot,y)\}\,\mathrm{d}x \\ + \int_\Omega \{\psi(\cdot,v) - v\,\mathrm{div}\,y + \psi^*(\cdot,\mathrm{div}\,y)\}\,\mathrm{d}x\,.\end{aligned}$$

Proof.

- For every $v \in \mathrm{dom}(I)$, we have that

$$I(v) = \int_\Omega \phi(\cdot,\nabla v)\,\mathrm{d}x + \int_\Omega \psi(\cdot,y)\,\mathrm{d}x\,.$$

- For every $y \in \mathrm{dom}(-D) \cap W_N^{p'}(\mathrm{div};\Omega)$, we have that

$$D(y) = -\int_\Omega \phi^*(\cdot,y)\,\mathrm{d}x - \int_\Omega \psi^*(\cdot,\mathrm{div}\,y)\,\mathrm{d}x\,.$$

- For every $v \in W_D^{1,p}(\Omega)$ and $y \in W_N^{p'}(\mathrm{div};\Omega)$, we have that

$$0 = \int_\Omega \nabla v \cdot y\,\mathrm{d}x + \int_\Omega v\,\mathrm{div}\,y\,\mathrm{d}x\,. \qquad \blacksquare$$

Primal-dual gap identity

Theorem: (integral representation of primal-dual gap estimator)

For every $v \in \mathrm{dom}(I)$ and $y \in \mathrm{dom}(-D) \cap W_N^{p'}(\mathrm{div};\Omega)$, it holds that

$$\begin{aligned}\eta^2_{\mathrm{gap}}(v,y) = &\int_\Omega \{\phi(\cdot,\nabla v) - \nabla v\cdot y + \phi^*(\cdot,y)\}\,\mathrm{d}x\\ &+ \int_\Omega \{\psi(\cdot,v) - v\,\mathrm{div}\,y + \psi^*(\cdot,\mathrm{div}\,y)\}\,\mathrm{d}x\,.\end{aligned}$$

- **Residual interpretation**: By the *Fenchel–Young inequality*, for every $v \in \mathrm{dom}(I)$ and $y \in \mathrm{dom}(-D) \cap W_N^{p'}(\mathrm{div};\Omega)$, it holds that

$$\begin{aligned}\phi(\cdot,\nabla v) - \nabla v\cdot y + \phi^*(\cdot,y) &\geq 0 \quad \text{a.e. in } \Omega\,,\\ \psi(\cdot,v) - v\,\mathrm{div}\,y + \psi^*(\cdot,\mathrm{div}\,y) &\geq 0 \quad \text{a.e. in } \Omega\,.\end{aligned}$$

In particular, for $v \in \mathrm{dom}(I)$ and $y \in \mathrm{dom}(-D) \cap W_N^{p'}(\mathrm{div};\Omega)$, it holds that

$$\left.\begin{aligned}&\phi(\cdot,\nabla v) - \nabla v\cdot y + \phi^*(\cdot,y) = 0 \quad \text{a.e. in } \Omega\,,\\ \&\quad &\psi(\cdot,v) - v\,\mathrm{div}\,y + \psi^*(\cdot,\mathrm{div}\,y) = 0 \quad \text{a.e. in } \Omega\,.\end{aligned}\right\} \Leftrightarrow \quad I(v) = D(y)$$

$$\Leftrightarrow \quad \left\{\begin{aligned}&I(v) = I(u)\,,\\ \&\quad &D(y) = D(z)\,.\end{aligned}\right.$$

Examples

Examples: Poisson problem

- **Primal-dual gap estimator:** For every $v \in \mathrm{dom}(I)$ and $y \in \mathrm{dom}(-D)$, it holds that

$$\begin{aligned}\eta^2_{\mathrm{gap}}(v,y) &\coloneqq \int_\Omega \left\{\frac{1}{2}|\nabla v|^2 - \nabla v\cdot y + \frac{1}{2}|y|^2\right\}\mathrm{d}x \\ &\quad + \int_\Omega \left\{-f\,v - \cancel{v\,\underbrace{\operatorname{div} y}_{=-f}} - \cancel{\underbrace{I^{(\cdot)}_{\{-f\}}(\operatorname{div} y)}_{=0}}\right\}\mathrm{d}x \\ &= \underbrace{\frac{1}{2}\int_\Omega |\nabla v - y|^2\,\mathrm{d}x}_{=0 \;\Leftrightarrow\; y=\nabla v \quad \text{a.e. in }\Omega}\,.\end{aligned}$$

- **Primal-dual total error:** For every $v \in \mathrm{dom}(I)$ and $y \in \mathrm{dom}(-D)$, it holds that

$$\rho^2_{\mathrm{tot}}(v,y) = \frac{1}{2}\int_\Omega |\nabla v - \nabla u|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |y - z|^2\,\mathrm{d}x\,.$$

- **Primal-dual gap identity:** For every $v \in \mathrm{dom}(I)$ and $y \in \mathrm{dom}(-D)$, it holds that

$$\frac{1}{2}\int_\Omega |\nabla v - \nabla u|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |y - z|^2\,\mathrm{d}x = \frac{1}{2}\int_\Omega |\nabla v - y|^2\,\mathrm{d}x\,,$$

i.e., the *Prager–Synge–Mikhlin identity*!

Examples: p-Dirichlet problem

- **Primal-dual gap estimator:** For every $v \in \mathrm{dom}(I)$ and $y \in \mathrm{dom}(-D)$, it holds that

$$\begin{aligned}\eta^2_{\mathrm{gap}}(v,y) &\coloneqq \int_\Omega \Big\{\frac{1}{p}|\nabla v|^p - \nabla v\cdot y + \frac{1}{p'}|y|^{p'}\Big\}\,\mathrm{d}x \\ &\quad + \int_\Omega \Big\{-f\,v - \cancel{v\underbrace{\operatorname{div} y}_{=-f}} - \cancel{\underbrace{I^{(\cdot)}_{\{-f\}}(\operatorname{div} y)}_{=0}}\Big\}\,\mathrm{d}x \\ &= \underbrace{\int_\Omega \Big\{\frac{1}{p}|\nabla v|^p - \nabla v\cdot y + \frac{1}{p'}|y|^{p'}\Big\}\,\mathrm{d}x}_{=0\quad\Leftrightarrow\quad y=|\nabla v|^{p-2}\nabla v\quad\text{a.e. in }\Omega}\,.\end{aligned}$$

- **Primal-dual total error:** For every $v \in \mathrm{dom}(I)$ and $y \in \mathrm{dom}(-D)$, it holds that

$$\rho^2_{\mathrm{tot}}(v,y) \sim \int_\Omega |V(\nabla v) - V(\nabla u)|^2\,\mathrm{d}x + \int_\Omega |V^*(y) - V^*(z)|^2\,\mathrm{d}x\,.$$

- **Primal-dual gap equivalence:** For every $v \in \mathrm{dom}(I)$ and $y \in \mathrm{dom}(-D)$, it holds that

$$\begin{aligned}&\int_\Omega |V(\nabla v) - V(\nabla u)|^2\,\mathrm{d}x + \int_\Omega |V^*(y) - V^*(z)|^2\,\mathrm{d}x \\ &\qquad\sim \int_\Omega \Big\{\frac{1}{p}|\nabla v|^p - \nabla v\cdot y + \frac{1}{p'}|y|^{p'}\Big\}\,\mathrm{d}x\,.\end{aligned}$$

Examples: Obstacle problem

- **Primal-dual gap estimator:** For every $v \in \operatorname{dom}(I)$ and $y \in \operatorname{dom}(-D)$, it holds that

$$\begin{aligned}
\eta^2_{\mathrm{gap}}(v,y) &\coloneqq \int_\Omega \Big\{ \frac{1}{2}|\nabla v|^2 - \nabla v \cdot y + \frac{1}{2}|y|^2 \Big\}\,\mathrm{d}x \\
&\quad - \int_\Omega \Big\{ f\,v - v\,\operatorname{div} y - \underbrace{\cancel{I^{(\cdot)}_{\leq 0}(f + \operatorname{div} y)}}_{=0} \Big\}\,\mathrm{d}x \\
&= \frac{1}{2}\int_\Omega |\nabla v - y|^2\,\mathrm{d}x \underbrace{- \int_\Omega (f + \operatorname{div} y)\,v\,\mathrm{d}x}_{=0 \quad \Leftrightarrow \quad (f + \operatorname{div} y)\,v = 0 \quad \text{a.e. in } \Omega\,.}\,.
\end{aligned}$$

- **Primal-dual total error:** For every $v \in \operatorname{dom}(I)$ and $y \in \operatorname{dom}(-D)$, it holds that

$$\begin{aligned}
\rho^2_{\mathrm{tot}}(v,y) &= \frac{1}{2}\int_\Omega |\nabla v - \nabla u|^2\,\mathrm{d}x - \langle f + \operatorname{div} z, v\rangle_{W^{1,2}_D(\Omega)} \\
&\quad + \frac{1}{2}\int_\Omega |y - z|^2\,\mathrm{d}x - \int_\Omega (f + \operatorname{div} y)\,u\,\mathrm{d}x\,.
\end{aligned}$$

- **Primal-dual gap identity:** For every $v \in \operatorname{dom}(I)$ and $y \in \operatorname{dom}(-D)$, it holds that

$$\begin{aligned}
\frac{1}{2}\int_\Omega |\nabla v - \nabla u|^2\,\mathrm{d}x - \langle f + \operatorname{div} z, v\rangle_{W^{1,2}_D(\Omega)} + \frac{1}{2}\int_\Omega |y - z|^2\,\mathrm{d}x - \int_\Omega (f + \operatorname{div} y)\,u\,\mathrm{d}x \\
= \frac{1}{2}\int_\Omega |\nabla v - y|^2\,\mathrm{d}x - \int_\Omega (f + \operatorname{div} y)\,v\,\mathrm{d}x\,.
\end{aligned}$$

Major weakness: Practicability

- **Major weakness:** Which $(v,y)^\top \in \mathrm{dom}(I) \times (\mathrm{dom}(-D) \cap W_N^{p'}(\mathrm{div};\Omega))$ in

$$\rho_{I,\mathrm{opt}}^2(v,u) + \rho_{-D,\mathrm{opt}}^2(y,z) = \eta_{\mathrm{gap}}^2(v,y)\,?$$

- **Related contributions:**

→ **Equilibrated flux reconstruction:**

- W. Prager, J. Synge, *Approximations in elasticity based on the concept of function space*, **Quart. Appl. Math.**, 1947.
- R. Bank, A. Weiser, *Some a posteriori error estimators for elliptic partial differential equations*, **Math. Comp.**, 1985.
- D. Braess, J. Schöberl, *Equilibrated residual error estimator for edge elements*, **Math. Comp.**, 1993.
- M. Ainsworth, J. Oden, *A posteriori error estimation in finite element analysis*, **Wiley, New York**, 2000.

$$\begin{aligned}
y &= y_h^\Delta + \nabla u_h\,, && \text{where}\\
y_h^\Delta &= \sum_{\nu\in\mathcal{N}_h} y_\nu^\Delta\,, && \text{where } y_\nu^\Delta \text{ solves}\\
\mathrm{div}\, y_\nu^\Delta|_T &= -\tfrac{1}{|T|}(f,\varphi_\nu)_T && \text{for all } T\subseteq\omega_\nu\,,\\
[\![y_\nu^\Delta\cdot n]\!]_S &= -\tfrac{1}{d}[\![\nabla u_h\cdot n]\!]_S && \text{for all } S\subseteq\omega_\nu\,,\\
[\![y_\nu^\Delta\cdot n]\!] &= 0 && \text{on } \partial\omega_\nu\,,
\end{aligned}$$

→ **Averaging flux reconstruction:**

- O. Zienkiewicz, J. Zhu, *A simple error estimator and adaptive procedure for practical engineering analysis*, **Internat. J. Numer. Meth. Engrg.**, 1987.
- S. Bartels, C. Carstensen, *Each averaging technique yields reliable a posteriori error control in FEM on unstructured grids. Part I & Part II*, **Math. Comp.**, 2002.

$$\begin{aligned}
y &= \mathcal{A}_h(\nabla u_h)\,, && \text{where}\\
\mathcal{A}_h g &\coloneqq \sum_{\nu\in\mathcal{N}_h\setminus\Gamma_D} g_\nu\varphi_\nu\,, && \text{where}\\
g_\nu &\coloneqq \frac{(g,\psi_\nu)_{\mathrm{supp}(\psi_\nu)}}{(1,\varphi_\nu)_{\mathrm{supp}(\psi_\nu)}}
\end{aligned}$$

- **Discrete reconstruction formula:** discrete analogue of $z = D_t\phi(\cdot,\nabla u)$, *i.e.*,

$$y = z_h^{rt} = D_t\phi_h(\cdot,\nabla_h u_h^{cr}) + [\text{correction terms}]\,.$$

Thank you!

R. T. Rockafellar.
Integral functionals, normal integrands and measurable selections.
In J. P. Gossez, E. J. Lami Dozo, J. Mawhin, and L. Waelbroeck, editors, Nonlinear Operators and the Calculus of Variations, pages 157–207, Berlin, Heidelberg, 1976. Springer Berlin Heidelberg.