A priori and *a posteriori* error identities for convex minimization problems based on convex duality relations

Lecture 1

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"Oberseminar" of the Department of Applied Mathematics,
University of Freiburg,
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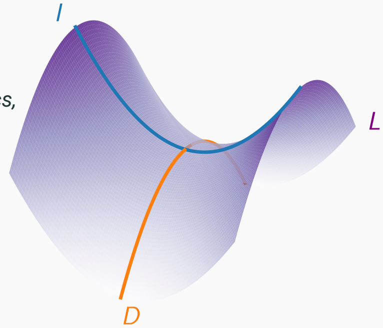
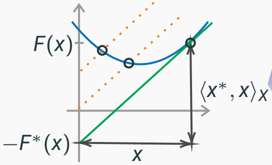
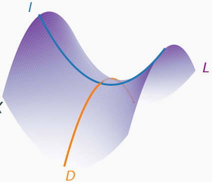


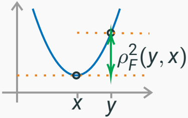
General objective

1. A posteriori error identities on the basis of <u>continuous</u> convex duality.





$$\rho^2_{\mathrm{tot}}(v,y) = \eta^2_{\mathrm{gap}}(v,y)$$



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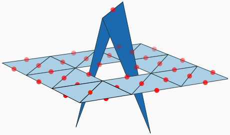
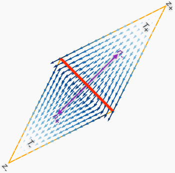
2. <u>Numerically practicable</u> a posteriori error identities on the basis of <u>discrete</u> convex duality.





$$z_h^{rt} = D_t\phi_h(\cdot, \nabla_h u_h^{cr}) + \frac{D_t\psi_h(\cdot, \Pi_h u_h^{cr})}{d}\left(\mathrm{id}_{\mathbb{R}^d} - \Pi_h \mathrm{id}_{\mathbb{R}^d}\right)$$

A priori error analysis
and
a posteriori error analysis

Concepts of *a priori* and *a posteriori* error analysis

- **Continuous problem:** Seek $u \in X$ s.t.

$$I(u) = \inf_{v \in X} I(v)\,,$$

where $I\colon X \to \mathbb{R} \cup \{+\infty\}$.

- **Discrete problem:** Seek $u_h \in X_h$ ($\not\subseteq X$) s.t.

$$I_h(u_h) = \inf_{v_h \in X_h} I_h(v_h)\,,$$

where $I_h \colon X_h \to \mathbb{R} \cup \{+\infty\}$.

- ***A priori* error estimate:** (continuous into discrete)

$$\rho_{I_h}^2(u_h, \Pi_h^{X_h} u) \leq \eta_{I_h}^2(\Pi_h^{X_h} u) = \begin{pmatrix}\text{independent of } u_h \\ \text{dependent of } u\end{pmatrix},$$

where

- $\rho_{I_h}^2 \colon X_h \times X_h \to \mathbb{R}_{\geq 0}$ is a *discrete distance measure*;
- $\Pi_h^{X_h} \colon X \to X_h$ a *(quasi-)interpolation operator*;
- $\eta_{I_h}^2 \colon X_h \to \mathbb{R}_{\geq 0}$ an *a priori error estimator*.

Concepts of *a priori* and *a posteriori* error analysis

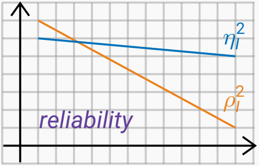
- ***A posteriori* error estimate:** (discrete into continuous)

$$\rho_I^2(u,\Pi_h^X u_h) \leq \eta_I^2(\Pi_h^X u_h) = \begin{pmatrix}\text{independent of } u\\ \text{dependent of } u_h\end{pmatrix},$$

  where
  - $\rho_I^2 : X \times X \to \mathbb{R}_{\geq 0}$ is a *distance measure*;
  - $\Pi_h^X : X_h \to X$ a *post-processing operator*;
  - $\eta_I^2 : X \to \mathbb{R}_{\geq 0}$ an *a posteriori error estimator*.

- **Reliability/Efficiency:** An *a posteriori* error estimator is called[1]
  - *reliable*, if

$$\rho_I^2(u,\Pi_h^X u_h) \lesssim \eta_I^2(\Pi_h^X u_h)\,;$$



[1] We write $A \lesssim B$ if $A \leq c\,B$ for a constant that does not depend on $h > 0$.

Concepts of *a priori* and *a posteriori* error analysis

- ***A posteriori* error estimate:** (discrete into continuous)

$$\rho_I^2(u, \Pi_h^X u_h) \leq \eta_I^2(\Pi_h^X u_h) = \begin{pmatrix} \text{independent of } u \\ \text{dependent of } u_h \end{pmatrix},$$

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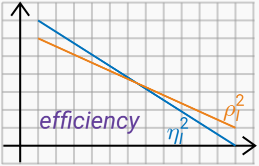
  - $\eta_I^2 : X \to \mathbb{R}_{\geq 0}$ an *a posteriori error estimator*.

- **Reliability/Efficiency:** An *a posteriori* error estimator is called[2]
  - *reliable*, if
  $$\rho_I^2(u, \Pi_h^X u_h) \lesssim \eta_I^2(\Pi_h^X u_h)\,;$$
  - *efficient*, if
  $$\rho_I^2(u, \Pi_h^X u_h) \gtrsim \eta_I^2(\Pi_h^X u_h)\,.$$



[2] We write $A \lesssim B$ if $A \leq c\,B$ for a constant that does not depend on $h > 0$.

Concepts of *a priori* and *a posteriori* error analysis

- ***A posteriori* error estimate:** (discrete into continuous)

$$\rho_I^2(u,\Pi_h^X u_h) \leq \eta_I^2(\Pi_h^X u_h) = \begin{pmatrix}\text{independent of } u\\ \text{dependent of } u_h\end{pmatrix},$$

  where
  - $\rho_I^2\colon X\times X\to\mathbb{R}_{\geq 0}$ is a *distance measure*;
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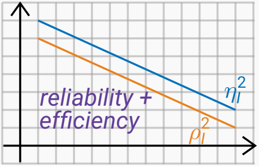
  - $\eta_I^2\colon X\to\mathbb{R}_{\geq 0}$ an *a posteriori error estimator*.

- **Reliability/Efficiency:** An *a posteriori* error estimator is called[3]
  - *reliable*, if
$$\rho_I^2(u,\Pi_h^X u_h) \lesssim \eta_I^2(\Pi_h^X u_h)\,;$$
  - *efficient*, if
$$\rho_I^2(u,\Pi_h^X u_h) \gtrsim \eta_I^2(\Pi_h^X u_h)\,.$$



- **Objective:** *A posteriori* error estimator $\eta_I^2\colon X\to\mathbb{R}_{\geq 0}$ that is *reliable* and *efficient*, *i.e.*,
$$\rho_I^2(u,\Pi_h^X u_h) \sim \eta_I^2(\Pi_h^X u_h)\,.$$

[3] We write $A\lesssim B$ if $A\leq c\,B$ for a constant that does not depend on $h>0$.

Concepts of *a priori* and *a posteriori* error analysis

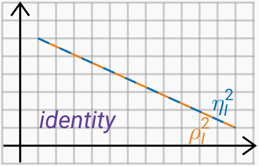
- ***A posteriori* error estimate:** (discrete into continuous)

$$\rho_I^2(u,\Pi_h^X u_h) \leq \eta_I^2(\Pi_h^X u_h) = \begin{pmatrix}\text{independent of } u\\ \text{dependent of } u_h\end{pmatrix},$$

where

- $\rho_I^2\colon X\times X\to\mathbb{R}_{\geq 0}$ is a *distance measure*;
- $\Pi_h^X\colon X_h\to X$ a *post-processing operator*;
- $\eta_I^2\colon X\to\mathbb{R}_{\geq 0}$ an *a posteriori error estimator*.

- **Reliability/Efficiency:** An *a posteriori* error estimator is called[4]
  - *reliable*, if
$$\rho_I^2(u,\Pi_h^X u_h) \lesssim \eta_I^2(\Pi_h^X u_h)\,;$$
  - *efficient*, if
$$\rho_I^2(u,\Pi_h^X u_h) \gtrsim \eta_I^2(\Pi_h^X u_h)\,.$$



- **Objective:** *A posteriori* error estimator $\eta_I^2\colon X\to\mathbb{R}_{\geq 0}$ that is *identical*, *i.e.*,

$$\rho_I^2(u,\Pi_h^X u_h) = \eta_I^2(\Pi_h^X u_h)\,.$$

[4] We write $A\lesssim B$ if $A\leq c\,B$ for a constant that does not depend on $h>0$.

Content of the course

- **Lecture 1: General convex duality theory**
- **Lecture 2: Convex duality theory for integral functionals**
- **Lecture 3: Convex duality theory for discrete integral functionals**
- **Lecture 4: Applications**

Content of the Lecture 1

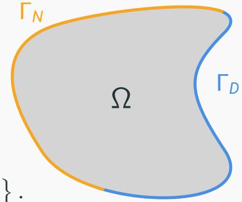
- **Lecture 1: General convex duality theory**
  - The Prager–Synge–Mikhlin identity.
  - Convex analysis.
  - Lagrange duality theory.
  - Fenchel duality theory.

The Prager–Synge–Mikhlin identity

Function spaces

- **Dirichlet/Neumann boundary:** Let $\Omega \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded Lipschitz domain and

$$\Gamma_D, \Gamma_N \subseteq \partial\Omega \quad \text{s.t.} \quad \Gamma_D \dot{\cup} \Gamma_N = \partial\Omega\,.$$



- **Sobolev spaces:** For $p \in [1, +\infty)$, we define

$$W^{1,p}(\Omega) \coloneqq \{v \in L^p(\Omega) \mid \nabla v \in (L^p(\Omega))^d\}\,,$$
$$W^{1,p}_D(\Omega) \coloneqq \{v \in W^{1,p}(\Omega) \mid v = 0 \text{ a.e. on } \Gamma_D\}\,.$$

- H(**div**)**-spaces:** For $p \in (1, +\infty]$, we define

$$W^{p'}(\mathrm{div};\Omega) \coloneqq \{y \in (L^{p'}(\Omega))^d \mid \mathrm{div}\, y \in L^{p'}(\Omega)\}\,,$$
$$W^{p'}_N(\mathrm{div};\Omega) \coloneqq \{y \in W^{p'}(\mathrm{div};\Omega) \mid \langle y\cdot n, v\rangle_{W^{1-\frac{1}{p},p}(\partial\Omega)} = 0 \text{ for all } v \in W^{1,p}_D(\Omega)\}\,,$$

where $(y \mapsto y\cdot n)\colon W^{p'}(\mathrm{div};\Omega) \to (W^{1-\frac{1}{p},p}(\partial\Omega))^*$, for every $v \in W^{1,p}(\Omega)$ defined by

$$\langle y\cdot n, v\rangle_{W^{1-\frac{1}{p},p}(\partial\Omega)} = \int_\Omega y\cdot\nabla v\,\mathrm{d}x + \int_\Omega \mathrm{div}\, y\, v\,\mathrm{d}x\,,$$

is the *normal trace operator*.

- **Statement**: *A posteriori* error identity for both
  - ✓ weak formulation;
  - ✓ mixed formulation;

  based on geometric argument.


Figure: LEFT: William Prager (23 May 1903 – 17 March 1980); RIGHT: John Lighton Synge (23 March 1897 – 30 March 1995).

- **Weak formulation**: Seek $u \in W^{1,2}_D(\Omega)$ s.t. for every $v \in W^{1,2}_D(\Omega)$, it holds that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f\, v \,\mathrm{d}x\,. \qquad (f \in L^2(\Omega))$$

- **Mixed formulation**: Seek $(z,u)^\top \in W^2_N(\mathrm{div};\Omega) \times L^2(\Omega)$ s.t. for every $(y,v)^\top \in W^2_N(\mathrm{div};\Omega) \times L^2(\Omega)$, it holds that

$$\begin{aligned} \int_\Omega z \cdot y \,\mathrm{d}x + \int_\Omega u \,\mathrm{div}\, y \,\mathrm{d}x &= 0 && (\Leftrightarrow \quad z = \nabla u)\,, \\ \int_\Omega \mathrm{div}\, z\, v \,\mathrm{d}x &= -\int_\Omega f\, v\,\mathrm{d}x && (\Leftrightarrow \quad \mathrm{div}\, z = -f)\,. \end{aligned} \qquad (f \in L^2(\Omega))$$

Prager–Synge identity

Theorem: (Prager–Synge identity, *cf.* [3, Prager & Synge, '47])

For every $v \in W^{1,2}_D(\Omega)$ and $y \in W^2_N(\mathrm{div} = -f; \Omega)$, it holds that

$$\underbrace{\frac{1}{2}\int_\Omega |\nabla v - \nabla u|^2 \,\mathrm{d}x}_{\text{(primal error)}} + \underbrace{\frac{1}{2}\int_\Omega |y - z|^2 \,\mathrm{d}x}_{\text{(dual error)}} = \underbrace{\frac{1}{2}\int_\Omega |\nabla v - y|^2 \,\mathrm{d}x}_{\text{(primal-dual gap)}} .$$

- **Proof (based on geometric argument).**
  - **Orthogonality relation:** Due to

$$\int_\Omega (y - z)\cdot(\nabla v - \nabla u)\,\mathrm{d}x = \int_\Omega (\underbrace{\mathrm{div}\, z}_{=-f} - \underbrace{\mathrm{div}\, y}_{=-f})\cdot(v - u)\,\mathrm{d}x = 0\,,$$

we have that

$$y - z \perp \nabla v - \nabla u \quad \text{in } (L^2(\Omega))^d\,.$$

$\Rightarrow$ Using the *Pythagoras theorem*, we conclude that

$$\int_\Omega |\nabla v - \nabla u|^2\,\mathrm{d}x + \int_\Omega |y - z|^2\,\mathrm{d}x = \int_\Omega |\nabla v \underbrace{- \nabla u + z}_{=0} - y|^2\,\mathrm{d}x$$

$$= \int_\Omega |\nabla v - y|^2\,\mathrm{d}x\,.$$

■

Mikhlin estimate

- **Statement**: *A posteriori* error estimate for only
  - ✓ primal problem;
  - ✗ dual problem;

  based on convex duality argument.

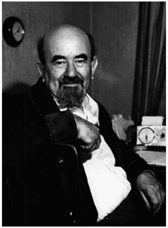


Figure: Solomon Grigorjewitsch Mikhlin (23 April 1908 – 29 August 1990).

- **Primal problem**: Seek $u \in W^{1,2}_D(\Omega)$ minimal for $I\colon W^{1,2}_D(\Omega) \to \mathbb{R}$, for every $v \in W^{1,2}_D(\Omega)$ defined by

$$I(v) \coloneqq \frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x - \int_\Omega f\,v\,\mathrm{d}x\,. \qquad (f \in L^2(\Omega))$$

- **Dual problem**: Seek $z \in W^2_N(\mathrm{div} = -f;\Omega)$ maximal for $D\colon W^2_N(\mathrm{div} = -f;\Omega) \to \mathbb{R}$, for every $y \in W^2_N(\mathrm{div} = -f;\Omega)$ defined by

$$D(y) \coloneqq -\frac{1}{2}\int_\Omega |y|^2\,\mathrm{d}x\,. \qquad (f \in L^2(\Omega))$$

Mikhlin estimate

Theorem: (Mikhlin estimate, *cf.* [2, Mikhlin, '64])

For every $v \in W^{1,2}_D(\Omega)$ and $y \in W^2_N(\mathrm{div} = -f; \Omega)$, it holds that

$$\frac{1}{2}\int_\Omega |\nabla v - \nabla u|^2 \,\mathrm{d}x \leq \frac{1}{2}\int_\Omega |\nabla v - y|^2 \,\mathrm{d}x\,.$$

- **Proof (based on convex duality argument).**
  - **Ingredient 1: (strong convexity)**

$$\frac{1}{2}\int_\Omega |\nabla v - \nabla u|^2 \,\mathrm{d}x \leq I(v) - I(u)\,.$$

  - **Ingredient 2: (weak duality)**

$$I(u) \geq D(y)\,.$$

$\Rightarrow$ Using the *integration-by-parts formula*, we conclude that

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla v - \nabla u|^2 \,\mathrm{d}x &\leq I(v) - D(y)\\ &= \frac{1}{2}\int_\Omega |\nabla v|^2 \,\mathrm{d}x - \int_\Omega \overbrace{f}^{= -\mathrm{div}\, y} v \,\mathrm{d}x + \frac{1}{2}\int_\Omega |y|^2 \,\mathrm{d}x\\ &= \frac{1}{2}\int_\Omega |\nabla v|^2 \,\mathrm{d}x - \int_\Omega \nabla v \cdot y \,\mathrm{d}x + \frac{1}{2}\int_\Omega |y|^2 \,\mathrm{d}x\,.\end{aligned}$$

∎

Theorem: (Prager–Synge–Mikhlin identity)

For every $v \in W^{1,2}_D(\Omega)$ and $y \in W^2_N(\mathrm{div} = -f; \Omega)$, it holds that

$$\frac{1}{2}\int_\Omega |\nabla v - \nabla u|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |y - z|^2 \,\mathrm{d}x = \frac{1}{2}\int_\Omega |\nabla v - y|^2 \,\mathrm{d}x \,.$$

- **Proof (based on convex duality argument).**
  - **Ingredient 1: (optimal strong convexity)**

$$\frac{1}{2}\int_\Omega |\nabla v - \nabla u|^2 \,\mathrm{d}x = I(v) - I(u)\,,$$

$$\frac{1}{2}\int_\Omega |y - z|^2 \,\mathrm{d}x = D(z) - D(y)\,.$$

  - **Ingredient 2: (strong duality)**

$$I(u) = D(z)\,.$$

⇒ Using the *integration-by-parts formula*, we conclude that

$$\frac{1}{2}\int_\Omega |\nabla v - \nabla u|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |y - z|^2 \,\mathrm{d}x = I(v) - \underbrace{[I(u) - D(z)]}_{=0} - D(y)$$

$$= I(v) - D(y)\,.$$

∎

Convex analysis

Convex, proper, and lower semi-continuous functionals

Definition: (convex functions)

Let X be a Banach space. Then, a functional $F\colon X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is called

(i) *convex*, if for every $x, y \in X$ and $\lambda \in [0, 1]$, it holds that

$$F(\lambda x + (1-\lambda) y) \leq \lambda F(x) + (1-\lambda) F(y)\,,$$

whenever the right-hand side is well-defined, *i.e.*, $\infty - \infty$ does not occur.

(ii) *proper*, if $F(x) > -\infty$ for all $x \in X$ and $\operatorname{dom} F \neq \emptyset$, where

$$\operatorname{dom} F \coloneqq \{x \in X \mid F(x) < \infty\}\,,$$

denotes the *effective domain* of F.

(iii) *lower semi-continuous or l.s.c.*, if for every $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $x \in X$, it holds that

$$x_n \to x \quad \text{in } X \quad (n \to \infty) \qquad \Rightarrow \qquad F(x) \leq \liminf_{n\to\infty} F(x_n)\,.$$

(iv) We define

$$\Gamma(X) \coloneqq \{F\colon X \to \overline{\mathbb{R}} \mid F \text{ convex, l.s.c., and } F > -\infty\} \,\dot{\cup}\, \{-\infty\}\,,$$
$$\Gamma_0(X) \coloneqq \{F \in \Gamma(X) \mid F \text{ proper}\}\,.$$

Fenchel conjugate

Definition: (Fenchel conjugate)

Let X be a Banach space and $F\colon X \to \overline{\mathbb{R}}$ a functional.

Then, the *Fenchel conjugate* $F^*: X^* \to \overline{\mathbb{R}}$, for every $x^* \in X^*$, is defined by

$$F^*(x^*) \coloneqq \sup_{x\in X} \left\{\langle x^*, x\rangle_X - F(x)\right\} .$$

- **Remarks:**
  - **Properties:** For every $x^* \in X^*$, it holds that

    $$F^*(x^*) = \sup_{x\in \operatorname{dom} F} \left\{\langle x^*, x\rangle_X - F(x)\right\},$$

    i.e., $F^* \in \Gamma(X^*)$;
  - **Fenchel–Young inequality:** For every $x \in X$ and $x^* \in X^*$, it holds that

    $$\langle x^*, x\rangle_X \leq F^*(x^*) + F(x) ,$$
    $$\text{with} \quad `{=}\text{'} \quad \Leftrightarrow \quad x^* \in \partial F(x) ,$$

    whenever $\infty - \infty$ does not occur.

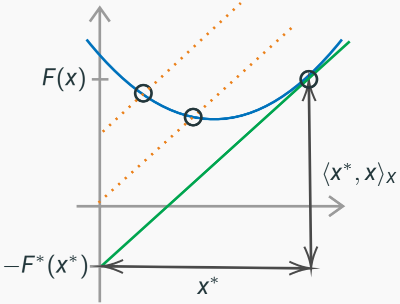


Figure: Geometric interpretation of Fenchel conjugate in 1D.

Sub-differential

Definition: (sub-differential)

Let X be a Banach space. Then, $F\colon X\to\overline{\mathbb{R}}$ is called *sub-differentiable* at $x\in X$ if $F(x)\neq\pm\infty$ and there exists $x^*\in X^*$, a so-called *sub-gradient*, s.t.

$$F(y)\geq F(x)+\langle x^*,y-x\rangle_X \quad \text{for all } y\in X\,. \qquad (*)$$

Then, the *sub-differential* $\partial F\colon X\to 2^{X^*}$, for every $x\in X$, is defined by

$$\partial F(x)\coloneqq\begin{cases}\emptyset & \text{if } F(x)=\pm\infty\,,\\ \{x^*\in X^*\mid (*) \text{ applies}\} & \text{else}\;.\end{cases}$$

- **Remarks:**
  - **Consistency:** If $F\colon X\to\mathbb{R}$ is convex and Gâteaux-differentiable at $x\in X$, then
    $$\partial F(x)=\{DF(x)\}\,.$$
  - **Optimality relation:** If $F\colon X\to\overline{\mathbb{R}}$ is proper, then
    $$F(x)=\inf_{y\in X}F(y)\quad\Leftrightarrow\quad 0\in\partial F(x)\,.$$

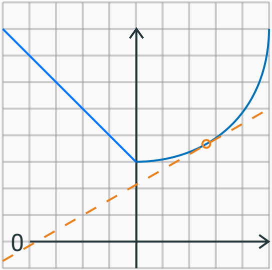


Figure: Geometric interpretation in 1D.

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- **Remarks:**
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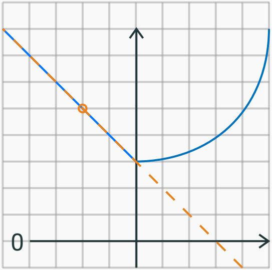


Figure: Geometric interpretation in 1D.

Definition: (sub-differential)

Let X be a Banach space. Then, $F\colon X \to \overline{\mathbb{R}}$ is called *sub-differentiable* at $x \in X$ if $F(x) \neq \pm\infty$ and there exists $x^* \in X^*$, a so-called *sub-gradient*, s.t.

$$F(y) \geq F(x) + \langle x^*, y - x\rangle_X \quad \text{for all } y \in X\,. \qquad (*)$$

Then, the *sub-differential* $\partial F\colon X \to 2^{X^*}$, for every $x \in X$, is defined by

$$\partial F(x) \coloneqq \begin{cases} \emptyset & \text{if } F(x) = \pm\infty\,, \\ \{x^* \in X^* \mid (*) \text{ applies}\} & \text{else}\,. \end{cases}$$

Remarks:

- **Consistency:** If $F\colon X \to \mathbb{R}$ is convex and Gâteaux-differentiable at $x \in X$, then

$$\partial F(x) = \{DF(x)\}\,.$$

- **Optimality relation:** If $F\colon X \to \overline{\mathbb{R}}$ is proper, then

$$F(x) = \inf_{y \in X} F(y) \quad \Leftrightarrow \quad 0 \in \partial F(x)\,.$$

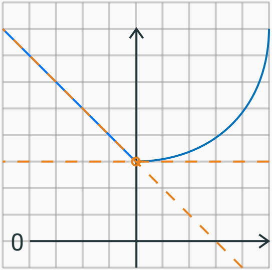


Figure: Geometric interpretation in 1D.

Sub-differential

Definition: (sub-differential)

Let X be a Banach space. Then, $F\colon X\to\overline{\mathbb{R}}$ is called *sub-differentiable* at $x\in X$ if $F(x)\neq\pm\infty$ and there exists $x^*\in X^*$, a so-called *sub-gradient*, s.t.

$$F(y)\geq F(x)+\langle x^*,y-x\rangle_X \quad \text{for all } y\in X\,. \qquad (*)$$

Then, the *sub-differential* $\partial F\colon X\to 2^{X^*}$, for every $x\in X$, is defined by

$$\partial F(x):=\begin{cases}\emptyset & \text{if } F(x)=\pm\infty\,,\\ \{x^*\in X^*\mid (*)\text{ applies}\} & \text{else}\ .\end{cases}$$

Remarks:

- **Consistency:** If $F\colon X\to\mathbb{R}$ is convex and Gâteaux-differentiable at $x\in X$, then

$$\partial F(x)=\{DF(x)\}\,.$$

- **Optimality relation:** If $F\colon X\to\overline{\mathbb{R}}$ is proper, then

$$F(x)=\inf_{y\in X}F(y) \quad\Leftrightarrow\quad 0\in\partial F(x)\,.$$

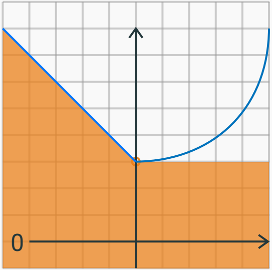


Figure: Geometric interpretation in 1D.

Definition: (bi-conjugate)

Let X be a Banach space and $F\colon X \to \overline{\mathbb{R}}$ a functional.

Then, the *Fenchel bi-conjugate* $F^{**}\colon X \to \overline{\mathbb{R}}$, for every $x \in X$, is defined by

$$F^{**}(x) \coloneqq \sup_{x^* \in X^*} \left\{ \langle x^*, x \rangle_{X^*} - F^*(x^*) \right\}.$$

- **Remarks:**
  - **Properties:** For every $x \in X$, it holds that

$$\begin{aligned} F^{**}(x) &= \sup_{x^* \in \operatorname{dom} F^*} \left\{ \langle x^*, x \rangle_X - F^*(x^*) \right\} \\ &\leq F(x)\,, \end{aligned}$$

    i.e., $F^{**} \in \Gamma(X)$;
  - **Fenchel–Moreau Theorem:** If $F \in \Gamma(X)$, then for every $x \in X$, we have that

$$F(x) = F^{**}(x) = \sup_{x^* \in X^*} \left\{ \langle x^*, x \rangle_{X^*} - F^*(x^*) \right\}.$$

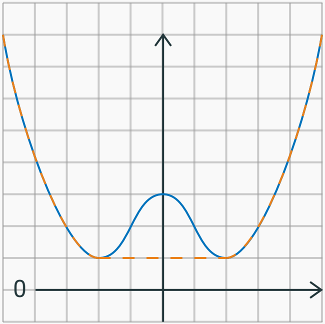


Figure: Double-well potential F (blue) and bi-conjugate F^{**} (orange).

Lagrange duality theory

Definition: (saddle point)

Let X, Y be Banach spaces and $L\colon X\times Y^*\to\overline{\mathbb{R}}$ a *Lagrange functional*.

A tuple $(u,z^*)^\top\in X\times Y^*$ is called *saddle point* of $L\colon X\times Y^*\to\overline{\mathbb{R}}$ if

$$\inf_{x\in X}\sup_{y^*\in Y^*} L(x,y^*) = L(u,z^*) = \sup_{y^*\in Y^*}\inf_{x\in X} L(x,y^*)\,.$$

- **Neumann's minimax theorem:**

(L.1) For every $y^*\in Y^*$, it holds that

$$L(\cdot,y^*)\in\Gamma_0(X)\,;$$

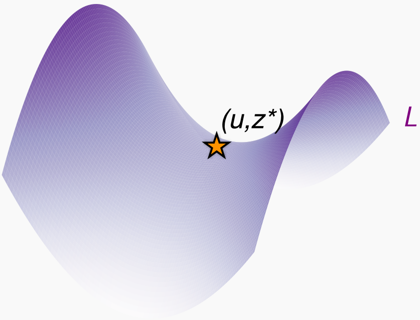
(L.2) For every $x\in X$, it holds that

$$-L(x,\cdot)\in\Gamma_0(Y^*)\,;$$

(L.3) There exists $y_0^*\in Y^*$ (or $x_0\in X$) s.t.

$$L(x,y_0^*)\to\infty\qquad(\|x\|_X\to\infty)\,,$$

$$(\text{or}\quad -L(x_0,y^*)\to\infty\qquad(\|y\|_{Y^*}\to\infty))\,.$$



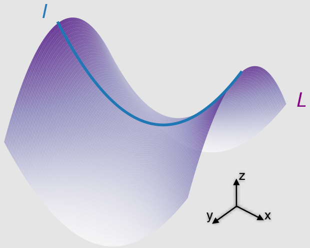
Definition: (primal problem & dual problem)

Let X, Y be Banach spaces and $L\colon X \times Y^* \to \overline{\mathbb{R}}$ a *Lagrange functional*.

- The *primal problem* is given via the minimization of $I\colon X \to \overline{\mathbb{R}}$, for every $x \in X$ defined by

$$I(x) \coloneqq \sup_{y^* \in Y^*} L(x, y^*)\,.$$

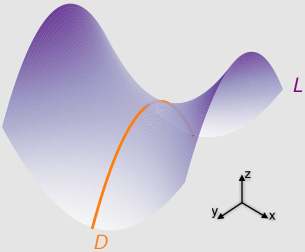
A minimizer $u \in X$ is called *primal solution*;



- The *dual problem* is given via the maximization of $D\colon Y^* \to \overline{\mathbb{R}}$, for every $y^* \in Y^*$ defined by

$$D(y^*) \coloneqq \inf_{x \in X} L(x, y^*)\,.$$

A maximizer $z^* \in Y^*$ is called *dual solution*.



Theorem: (weak duality & strong duality)

Let X, Y be Banach spaces and $L\colon X \times Y^* \to \overline{\mathbb{R}}$ a *Lagrange functional*.

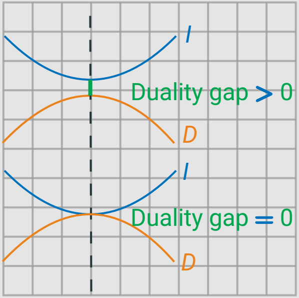
(i) A *weak duality relation* applies, *i.e.*,

$$\inf_{x\in X} I(x) \geq \sup_{y^*\in Y^*} D(y^*)\,;$$

(ii) A *strong duality relation* applies, *i.e.*,

$$I(u) = D(z^*)\,,$$

if and only if $(u,z^*)^\top \in X \times Y^*$ is a saddle point of $L\colon X \times Y^* \to \overline{\mathbb{R}}$.



♦ **Proof.**

ad (i).

$$\begin{aligned}\inf_{x\in X} I(x) &= \inf_{x\in X}\sup_{y^*\in Y^*} L(x,y^*)\\ &\geq \sup_{y^*\in Y^*}\inf_{x\in X} L(x,y^*)\\ &= \sup_{y^*\in Y^*} D(y^*)\,.\end{aligned}$$

$$\left(\sup_{y^*\in Y^*} L(x,y^*) \geq \sup_{y^*\in Y^*}\inf_{x\in X} L(x,y^*)\right)$$

Theorem: (weak duality & strong duality)

Let X, Y be Banach spaces and $L : X \times Y^* \to \overline{\mathbb{R}}$ a *Lagrange functional*.

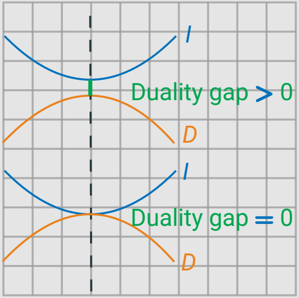
(i) A *weak duality relation* applies, *i.e.*,

$$\inf_{x\in X} I(x) \geq \sup_{y^*\in Y^*} D(y^*)\,;$$

(ii) A *strong duality relation* applies, *i.e.*,

$$I(u) = D(z^*)\,,$$

if and only if $(u, z^*)^\top \in X \times Y^*$ is a saddle point of $L : X \times Y^* \to \overline{\mathbb{R}}$.



Proof.

ad (ii). Due to **(i)**, it holds that

$$I(u) = D(z^*) \quad\Leftrightarrow\quad \sup_{y^*\in Y^*} L(u, y^*) = \inf_{x\in X} L(x, z^*)$$

$$\Leftrightarrow\quad \inf_{x\in X}\sup_{y^*\in Y^*} L(x, y^*) = L(u, z^*) = \sup_{y^*\in Y^*}\inf_{x\in X} L(x, y)\,. \quad ■$$

Fenchel duality theory

(Fenchel) primal problem

Definition: ((Fenchel) primal problem)

Let X, Y be Banach spaces, $F \in \Gamma_0(X)$, $G \in \Gamma_0(Y)$, and $\Lambda \in L(X;Y)$.

The *(Fenchel) primal problem* is given via the minimization of $I\colon X \to \mathbb{R} \cup \{+\infty\}$, for every $x \in X$ defined by

$$I(x) \coloneqq F(x) + G(\Lambda x)\,.$$

Example: Poisson problem

- $F \in \Gamma_0(W_D^{1,2}(\Omega))$, for every $v \in W_D^{1,2}(\Omega)$, defined by
$$F(v) \coloneqq -\int_\Omega f\, v \,\mathrm{d}x\,.$$
- $G \in \Gamma_0((L^2(\Omega))^d)$, for every $y \in (L^2(\Omega))^d$, defined by
$$G(y) \coloneqq \frac{1}{2}\int_\Omega |y|^2 \,\mathrm{d}x\,.$$
- $\Lambda \in L(W_D^{1,2}(\Omega);(L^2(\Omega))^d)$, for every $v \in W_D^{1,2}(\Omega)$, defined by
$$\Lambda v \coloneqq \nabla v\,.$$

Figure: Moritz Werner Fenchel (3 May 1905 – 24 January 1988).

Proposition: ((Fenchel) Lagrange functional)

Let X, Y be Banach spaces, $F \in \Gamma_0(X)$, $G \in \Gamma_0(Y)$, and $\Lambda \in L(X;Y)$.

Then, the *Lagrange functional* $L\colon X \times Y^* \to \overline{\mathbb{R}}$, for every $(x,y)^\top \in X \times Y^*$ defined by

$$L(x,y^*) \coloneqq F(x) + \langle y^*, \Lambda x\rangle_Y - G^*(y^*)\,,$$

induces the (Fenchel) primal problem, *i.e.*, for every $x \in X$, it holds that

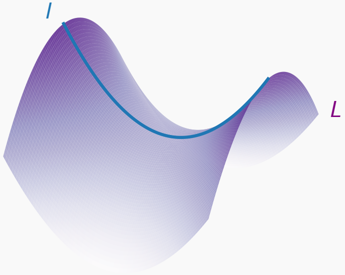
$$I(x) = \sup_{y^* \in Y^*} L(x,y^*)\,.$$

Proof. For every $y \in Y$, we have that

$$\begin{aligned} G(y) &= G^{**}(y) \\ &\coloneqq \sup_{y^* \in Y^*} \left\{\langle y^*, y\rangle_Y - G^*(y^*)\right\}, \end{aligned}$$

so that for every $x \in X$, it follows that

$$\begin{aligned} I(x) &\coloneqq F(x) + G(\Lambda x) \\ &= F(x) + \sup_{y^* \in Y^*} \left\{\langle y^*, \Lambda x\rangle_Y - G^*(y^*)\right\} \\ &= \sup_{y^* \in Y^*} \left\{F(x) + \langle y^*, \Lambda x\rangle_Y - G^*(y^*)\right\}. \blacksquare \end{aligned}$$



(Fenchel) dual problem

Proposition: ((Fenchel) dual problem)

Let X, Y be Banach spaces, $F \in \Gamma_0(X)$, $G \in \Gamma_0(Y)$, and $\Lambda \in L(X;Y)$.

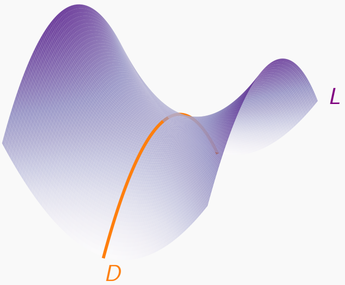
Then, the *(Fenchel) dual problem* is given via the maximization of $D\colon Y^*\to\mathbb{R}\cup\{-\infty\}$, for every $y^* \in Y^*$ defined by

$$D(y^*) \coloneqq -F^*(-\Lambda^* y^*) - G^*(y^*)\,.$$

- **Proof.** For every $y^* \in Y^*$, it holds that

$$\begin{aligned} D(y^*) &\coloneqq \inf_{x\in X} L(x,y^*) \\ &= \inf_{x\in X}\{F(x) + \langle y^*, \Lambda x\rangle_Y - G^*(y^*)\} \\ &= \inf_{x\in X}\{F(x) + \langle y^*, \Lambda x\rangle_Y\} - G^*(y^*) \\ &= \inf_{x\in X}\{F(x) + \langle \Lambda^* y^*, x\rangle_X\} - G^*(y^*) \\ &= -\sup_{x\in X}\{\langle -\Lambda^* y^*, x\rangle_X - F(x)\} - G^*(y^*) \\ &= -F^*(-\Lambda^* y^*) - G^*(y^*)\,. \end{aligned}$$

■



Lemma: (strong duality relation ⇔ convex optimality relations)

Let X, Y be Banach spaces, $F \in \Gamma_0(X)$, $G \in \Gamma_0(Y)$, and $\Lambda \in L(X;Y)$.

For $u \in X$ and $z^* \in Y^*$, the following statements are equivalent:

(i) A *strong duality relation* applies, *i.e.*, it holds that

$$I(u) = D(z^*)\,;$$

(ii) *Convex optimality relations* apply, *i.e.*, it holds that

$$G^*(z^*) - \langle z^*, \Lambda u\rangle_Y + G(\Lambda u) = 0\,,$$
$$F^*(-\Lambda^* z^*) - \langle -\Lambda^* z^*, u\rangle_X + F(u) = 0\,.$$

Proof. By the *Fenchel–Young inequality* and $\langle z^*, \Lambda u\rangle_Y = \langle \Lambda^* z^*, u\rangle_X$, it holds that

$$\text{(i)} \quad \Leftrightarrow \quad 0 = I(u) - D(z^*)\,.$$

$$\Leftrightarrow \quad 0 = \underbrace{\{G^*(z^*) - \langle z^*, \Lambda u\rangle_Y + G(\Lambda u)\}}_{\geq 0} + \underbrace{\{F^*(-\Lambda^* z^*) - \langle -\Lambda^* z^*, u\rangle_X + F(u)\}}_{\geq 0}\,.$$

$$\Leftrightarrow \quad \text{(ii)}\,.$$

■

Fenchel's duality theorem

Theorem: (Fenchel's duality theorem, *cf.* [1, Ekeland & Temam, '99])

Let X, Y be Banach spaces, $F \in \Gamma_0(X)$, $G \in \Gamma_0(Y)$, and $\Lambda \in L(X;Y)$.

If there exists $x_0 \in \mathrm{dom}(F)$ with $y_0 := \Lambda x_0 \in \mathrm{dom}(G)$ s.t.

$$y \to y_0 \quad \text{in } Y \qquad \Rightarrow \qquad G(y) \to G(y_0)\,,$$

then there exists a dual solution $z^* \in Y^*$ and

$$\inf_{v \in X} I(v) = D(z^*)\,.$$

Corollary: (*cf.* [1, Ekeland & Temam, '99])

Let the assumptions of Fenchel's duality theorem I be satisfied.

If X is reflexive and $I\colon X \to \mathbb{R} \cup \{+\infty\}$ is *(weakly) coercive*, *i.e.*,

$$\|x\|_X \to +\infty \qquad \Rightarrow \qquad I(x) \to +\infty\,,$$

then there exists a primal solution $u \in X$, a dual solution $z^* \in Y^*$, and

$$I(u) = D(z^*)\,.$$

Thank You for today!

Examples of Fenchel conjugates

Example: (p-Dirichlet density)

For $p \in (1, +\infty)$, let $F\colon X = \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$, for every $x \in \mathbb{R}^d$, be defined by

$$F(x) \coloneqq \frac{1}{p}|x|^p\,.$$

Then, $F^*\colon X^* \cong \mathbb{R}^d \to \mathbb{R}$, for every $x^* \in \mathbb{R}^d$, is given via

$$F^*(x^*) = \frac{1}{p'}|x^*|^{p'}\,. \qquad (p' = \tfrac{p}{p-1})$$

Proof. For every $x^* \in \mathbb{R}^d$, we have that

$$F^*(x^*) \coloneqq \sup_{x \in \mathbb{R}^d}\left\{x^* \cdot x - \frac{1}{p}|x|^p\right\}.$$

The supremum is attained at $x_0 \in X$ if

$$0 \overset{!}{=} \frac{\mathrm{d}}{\mathrm{d}x}\left\{x^* \cdot x - \frac{1}{p}|x|^p\right\}\Big|_{x=x_0} = x^* - |x_0|^{p-2}x_0\,.$$

$$\Leftrightarrow \quad x_0 = |x^*|^{p'-2}x^*\,.$$

Inserting $x_0 \in X$, we obtain

$$F^*(x^*) = x^* \cdot x_0 - \frac{1}{p}|x_0|^p = \frac{1}{p'}|x^*|^{p'}\,. \qquad \blacksquare$$

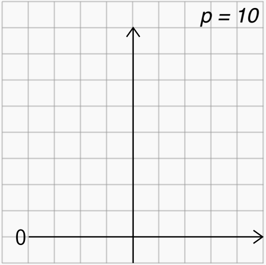


Figure: F (blue) and F^* (orange).

Example: (inequality constraint)

Let $F\colon X=\mathbb{R}\to\mathbb{R}$, for every $x\in\mathbb{R}$, be defined by

$$F(x) \coloneqq I_+(x) \coloneqq \begin{cases} 0 & \text{if } x\geq 0\,, \\ +\infty & \text{else}\,. \end{cases}$$

Then, $F^*\colon X^*\cong\mathbb{R}\to\mathbb{R}$, for every $x^*\in\mathbb{R}$, is given via

$$F^*(x^*) = I_-(x^*) \coloneqq \begin{cases} 0 & \text{if } x^*\leq 0\,, \\ +\infty & \text{else}\,. \end{cases}$$

Proof. For every $x^*\in\mathbb{R}$, we have that

$$\begin{aligned} F^*(x^*) &\coloneqq \sup_{x\in\mathbb{R}}\{x^*x - I_+(x)\} \\ &= \sup_{x\in\mathbb{R}_{\geq 0}} x^*x \\ &= \begin{cases} 0 & \text{if } x^*\leq 0\,, \\ +\infty & \text{else}\,. \end{cases} \\ &\eqqcolon I_-(x^*)\,. \end{aligned}$$

■

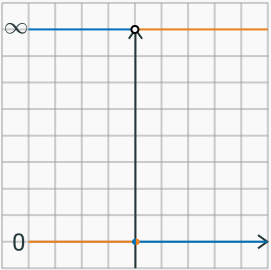


Figure: F (blue) and F^* (orange).

Example: (linear functional)

For $m \in \mathbb{R}$, let $F\colon X = \mathbb{R} \to \mathbb{R}$, for every $x \in \mathbb{R}$, be defined by

$$F(x) \coloneqq m\,x\,.$$

Then, $F^*\colon X^* \cong \mathbb{R} \to \mathbb{R}$, for every $x^* \in \mathbb{R}$, is given via

$$F^*(x^*) \coloneqq I_{\{m\}}(x^*) \coloneqq \begin{cases} 0 & \text{if } x^* = m\,, \\ +\infty & \text{else}\ . \end{cases}$$

Proof. For every $x^* \in \mathbb{R}$, we have that

$$\begin{aligned} F^*(x^*) &\coloneqq \sup_{x\in\mathbb{R}} \{x^*\,x - m\,x\} \\ &= \sup_{x\in\mathbb{R}} (x^* - m)\,x \\ &= \begin{cases} 0 & \text{if } x^* = m\,, \\ +\infty & \text{else}\ . \end{cases} \\ &\eqqcolon I_{\{m\}}(x^*)\,. \end{aligned}$$

■

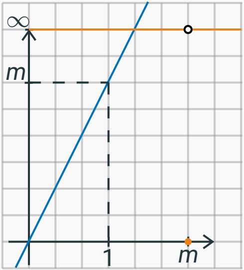


Figure: F (blue) and F^* (orange).

Literature recommendations

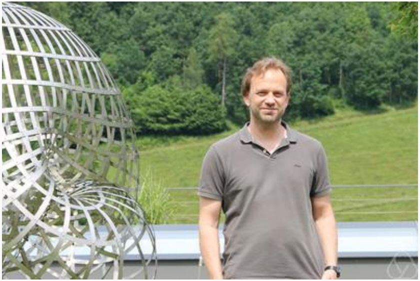


Figure: Sören Bartels

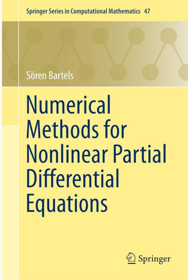


Figure: his book

Literature recommendations:

- S. Bartels, *Numerical methods for nonlinear partial differential equations*, **Springer Series in Computational Mathematics** 47, Springer, Cham, 2015. doi: 10.1007/978-3-319-13797-1.
- S. Bartels and M. Milicevic, *Primal-dual gap estimators for a posteriori error analysis of nonsmooth minimization problems*, **ESAIM Math. Model. Numer. Anal.** 54 (2020), 1635–1660.doi: 10.1051/m2an/2019074.
- S. Bartels and Z. Wang, *Orthogonality relations of Crouzeix-Raviart and Raviart-Thomas finite element spaces*, **Numer. Math.** 148 no. 1 (2021), 127–139. doi: 10.1007/s00211-021-01199-3.
- S. Bartels, *Nonconforming discretizations of convex minimization problems and precise relations to mixed methods*, **Comput. Math. Appl.** 93 (2021), 214–229. doi: 10.1016/j.camwa.2021.04.014.

Literature recommendations

Figure: Sergey I. Repin

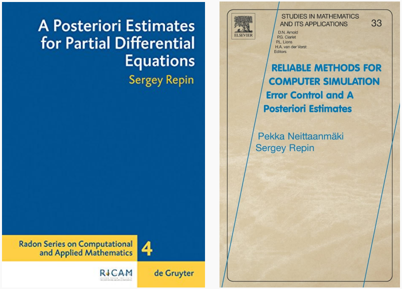


Figure: his books

Literature recommendations:

- S. I. Repin, *A posteriori error estimation for variational problems with uniformly convex functionals*, **Math. Comp.** 69 no. 230 (2000), 481–500. doi: 10.1090/S0025-5718-99-01190-4.
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