ABSCHLUSSBERICHT

1 Allgemeine Angaben

DFG-Geschäftszeichen: KA 6129/1-1:1

Projektnummer: 525389262

Titel des Projekts:

- Englischer Titel: *Numerical methods for physical problems with variable growth conditions*
- Deutscher Titel: *Numerische Methoden für physikalische Probleme mit variablen Wachstumsbedingungen*

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2 Summary

2.1 English summary

This research project aims at theoretical and experimental investigations on numerical methods for the physical models for *smart fluids*, *e.g.*, electro-rheological fluids (*cf.* [1, 2]), micro-polar electro-rheological fluids (*cf.* [3, 4]), magneto-rheological fluids (*cf.* [5]), chemically reacting fluids (*cf.* [6, 7]), and thermo-rheological fluids (*cf.* [8, 9]). Smart fluids have the potential for an application in numerous areas, *e.g.*, in electronic, automobile, heavy machinery, military, and biomedical industry (*cf.* [10], for an overview). All models mentioned above have a common characteristic: all models consist of coupled systems of elliptic or parabolic partial differential equations containing the generalized Navier–Stokes equations with a time- and space-dependent power-law index, the so-called *unsteady $p(t,x)$-Navier–Stokes equations*. The latter seek in a bounded domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, and over a time interval $I := (0,T)$, $0 < T < \infty$, abbre*via*ting $Q_T := I \times \Omega$ and $\Gamma_T := I \times \partial\Omega$, for a *velocity vector field* $\mathbf{v} \colon \overline{Q_T} \to \mathbb{R}^d$ and a *kinematic pressure* $q \colon Q_T \to \mathbb{R}$ such that

$$\begin{aligned} \partial_t \mathbf{v} - \operatorname{div}_x \mathbf{S}(\cdot,\cdot,\mathbf{D}_x\mathbf{v}) + \operatorname{div}_x(\mathbf{v}\otimes\mathbf{v}) + \nabla_x q &= \mathbf{f} && \text{in } Q_T\,, \\ \operatorname{div}_x \mathbf{v} &= 0 && \text{in } Q_T\,, \\ \mathbf{v} &= \mathbf{0} && \text{on } \Gamma_T\,, \\ \mathbf{v}(0) &= \mathbf{v}_0 && \text{in } \Omega\,, \end{aligned} \tag{1}$$

where $\mathbf{f} \colon Q_T \to \mathbb{R}^d$ describes external forces and $\mathbf{v}_0 \colon \Omega \to \mathbb{R}^d$ is the initial velocity vector field at time $t = 0$. In the system (1), the *extra stress tensor* $\mathbf{S} \colon Q_T \times \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ depends on the *strain rate tensor* $\mathbf{D}_x\mathbf{v} := \frac{1}{2}(\nabla_x\mathbf{v} + \nabla_x\mathbf{v}^\top) \colon Q_T \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and assumes for a.e. $(t,x)^\top \in Q_T$ and all $\mathbf{A} \in \mathbb{R}^{d\times d}$, *e.g.*, the form

$$\mathbf{S}(t,x,\mathbf{A}) := (\delta + |\mathbf{A}^{\mathrm{sym}}|)^{p(t,x)-2}\mathbf{A}^{\mathrm{sym}}\,,$$

where $\delta \geq 0$ and the *power-law index* $p \colon Q_T \to (1,+\infty)$ is at least (Lebesgue) measurable with

$$1 < p^- := \operatorname{ess\,inf}_{(t,x)^\top \in Q_T} p(t,x) \leq p^+ := \operatorname{ess\,sup}_{(t,x)^\top \in Q_T} p(t,x) < \infty\,.$$

The time- and space-dependency of the power-law index in (1) results in a number of mathematical hurdles; *e.g.*, the invalidity of a Poincaré inequality, a Korn inequality, and a negative norm inequality (*cf.* [11, 12]). Due to these hurdles, for two decades merely partial existence results could be established (*cf.* [13–16]). Consequently, only a few numerical investigations have been carried out (*cf.* [17–23]). In the PhD thesis [11], methods to overcome the hurdles mentioned above were developed and a thorough existence analysis for the unsteady $p(t,x)$-Navier–Stokes equations (1) was established. The methods from the PhD thesis [11], in the present research project, provide a firm foundation for a thorough numerical analysis: More precisely, the present research project comprises the following two mutually connected work areas:

Work Area A: Finite element (FE) approximation of the steady $p(x)$-Navier–Stokes equations.

Work Area B: Finite element (FE) approximation of the unsteady $p(t,x)$-Navier–Stokes equations.

2.2 German summary

Dieses Forschungsprojekt zielt auf theoretische und experimentelle Untersuchungen zu numerischen Methoden für die physikalischen Modelle von *intelligenten Flüssigkeiten* ab, z. B. elektro-rheologische Flüssigkeiten (*cf.* [1, 2]), mikro-polar elektro-rheologische Flüssigkeiten (*cf.* [3, 4]), magneto-rheologische Flüssigkeiten (*cf.* [5]), chemisch reagierende Flüssigkeiten (*cf.* [6, 7]) und thermo-rheologische Flüssigkeiten (*cf.* [8, 9]). Intelligente Flüssigkeiten haben das Potenzial für die Anwendung in zahlreichen Bereichen, z. B. in der Elektronik-, Automobil-, Schwermaschinen-, Militär- und biomedizinischen Industrie (*cf.* [10], für einen Überblick). Alle oben genannten Modelle haben ein gemeinsames Merkmal: alle Modelle bestehen aus gekoppelten Systemen elliptischer oder parabolischer partieller Differentialgleichungen, welche die verallgemeinerten Navier–Stokes-Gleichungen mit einem zeit- und ortsabhängigen Fließ-Index enthalten, die sogenannten *instationären $p(t,x)$-Navier–Stokes-Gleichungen*. Letztere suchen in einem beschränkten Gebiet $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, und in einem endlichen Zeitintervall $I := (0,T)$, $0 < T < \infty$, nach einem *Geschwindigkeitsvektorfeld* $\mathbf{v}\colon \overline{Q_T} \to \mathbb{R}^d$ und einem *kinematischen Druck* $q\colon Q_T \to \mathbb{R}$, sodass

$$\begin{aligned} \partial_t \mathbf{v} - \operatorname{div}_x \mathbf{S}(\cdot,\cdot,\mathbf{D}_x\mathbf{v}) + \operatorname{div}_x(\mathbf{v}\otimes\mathbf{v}) + \nabla_x q &= \mathbf{f} && \text{in } Q_T\,, \\ \operatorname{div}_x \mathbf{v} &= 0 && \text{in } Q_T\,, \\ \mathbf{v} &= \mathbf{0} && \text{auf } \Gamma_T\,, \\ \mathbf{v}(0) &= \mathbf{v}_0 && \text{in } \Omega\,, \end{aligned} \tag{2}$$

wobei $\mathbf{f}\colon Q_T \to \mathbb{R}^d$ äußere Kräfte beschreibt und $\mathbf{v}_0\colon \Omega \to \mathbb{R}^d$ die Anfangsgeschwindigkeit zum Zeitpunkt $t = 0$ ist. In dem System (2) hängt der *Extra-Stresstensor* $\mathbf{S}\colon Q_T \times \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ vom *Scheerratentensor* $\mathbf{D}_x\mathbf{v} := \frac{1}{2}(\nabla_x\mathbf{v} + \nabla_x\mathbf{v}^\top)\colon Q_T \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ ab und hat für f.a. $(t,x)^\top \in Q_T$ und alle $\mathbf{A} \in \mathbb{R}^{d\times d}$, z. B. die Form

$$\mathbf{S}(t,x,\mathbf{A}) := (\delta + |\mathbf{A}^{\mathrm{sym}}|)^{p(t,x)-2}\mathbf{A}^{\mathrm{sym}}\,,$$

wobei $\delta \geq 0$ und der *Fließ-Index* $p\colon Q_T \to (1,+\infty)$ mindestens (Lebesgue-)messbar ist mit

$$1 < p^- := \operatorname{ess\,inf}_{(t,x)^\top \in Q_T} p(t,x) \leq p^+ := \operatorname{ess\,sup}_{(t,x)^\top \in Q_T} p(t,x) < \infty\,.$$

Die Zeit- und Ortsabhängigkeit des Fließ-Indexes in (2) führt zu einer Reihe von mathematischen Hürden; z. B. die Ungültigkeit einer Poincaré-Ungleichung, einer Korn-Ungleichung und einer Negativen-Norm-Ungleichung (*cf.* [11, 12]). Wegen dieser Hürden konnten für zwei Jahrzehnte nur Teilexistenzresultate bewiesen werden (*cf.* [13–16]). Folglich wurden nur wenige numerische Untersuchungen durchgeführt (*cf.* [17–23]). In der Dissertation [11] wurden Methoden zur Überwindung der oben genannten Hürden entwickelt und eine vollständige Existenzanalyse für die instationären $p(t,x)$-Navier–Stokes-Gleichungen (2) durchgeführt. Die Methoden der Dissertation [11] bilden in diesem Forschungsprojekt die Grundlage für numerische Analysen. Genauer umfasst dieses Forschungsprojekt die folgenden zwei miteinander verbundenen Arbeitsbereiche:

Arbeitsbereich A: Finite-Elemente-(FE)-Approximation der stationären $p(x)$-Navier–Stokes-Gleichungen.

Arbeitsbereich B: Finite-Elemente-(FE)-Approximation der instationären $p(t,x)$-Navier–Stokes-Gleichungen.

3 Scientific work and results report

3.1 Initial objectives and results of the project

Work Area A: Finite element (FE) approximation of the steady $p(x)$-Navier–Stokes equations.

This work area aims at theoretical and experimental investigations on a finite element (FE) approximation of the steady $p(x)$-Navier–Stokes equations and is divided into three work packages:

Work Package A1: Weak convergence of a finite element (FE) approximation.

Objective A1: The objective of this work package is to establish a general framework that allows to conclude the well-posedness (*i.e.*, existence of discrete solutions), stability (*i.e.*, *a priori* estimates), and weak convergence of a FE approximation of a steady problem, containing the steady $p(x)$-Navier–Stokes equations as a particular case, for a possibly discontinuous power-law index $p \in L^\infty(\Omega)$ with $p^- > 1$ (or $p^- > \frac{3d}{d+2}$ if the convective term is present) and general approximations $(p_h)_{h>0} \subseteq L^\infty(\Omega)$ satisfying $p_h \to p$ $(h \to 0)$.

Result A1: This work package was substituted by Work Package B3, which was added to Work Area B in the course of this research project. However, the findings of Work Package B2 can straightforwardly be reduced to a finite element approximation of the steady $p(x)$-Navier–Stokes equations, so that Work Package A1 indirectly is already partially completed and will be completed in the future.

Work Package A2: Error estimates for a finite element (FE) approximation.

Objective A2.1: An objective of this work package is to derive *a priori* error estimates for a finite element approximation of the steady $p(x)$-Navier–Stokes equations, using Témam's modification (*cf.* [24]) to discretize the convective term and a one-point quadrature rule to discretize the power-law index, imposing slightly more resitrictive assumptions on the regularity of the kinematics pressure than $q \in W^{1,p'(\cdot)}(\Omega)$, *e.g.*, required in [21].

Result A2.1: In the case of an only Hölder continuous power-law index $p \in C^{0,\alpha}(\overline{\Omega})$, $\alpha \in (0,1]$, one cannot hope for the *"full"* regularity $\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) \in W^{1,2}(\Omega;\mathbb{R}^{d\times d})$ and $q \in W^{1,p'(\cdot)}(\Omega)$, but instead it is reasonable to expect (*cf.* [20, Rem. 4.5]) the *"partial"* regularity $\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) \in N^{\beta,2}(\Omega;\mathbb{R}^{d\times d})$, $\beta \in (0,1]$, where $N^{\beta,2}(\Omega;\mathbb{R}^{d\times d})$ denotes the Nikolskiĭ space (*cf.* [25]). Concerning the kinematic pressure, it is proposed to consider, *e.g.*, the regularity $q \in H^{\gamma,p'(\cdot)}(\Omega)$, $\gamma \in (0,1]$, where $H^{\gamma,p'(\cdot)}(\Omega)$ denotes the fractional variable Hajłasz–Sobolev space (*cf.* [26, 27]). If $p \in C^{0,\alpha}(\overline{\Omega})$, $\alpha \in (0,1]$, with $p^- \geq \frac{3d}{d+2}$, $\delta > 0$, $\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) \in N^{\beta,2}(\Omega;\mathbb{R}^{d\times d})$, $\beta \in (0,1]$, and $q \in H^{\gamma,p'(\cdot)}(\Omega)$, $\gamma \in (\frac{\alpha}{\min\{2,(p^+)'\}},1]$, it was established (*cf.* [28, Cor. 6.2(6.3)]) that there exists a constant $s > 1$, which can chosen to be close to 1 if $h > 0$ is close to 0, and a constant $c_0 > 0$ such that if $\|\mathbf{D}\mathbf{v}\|_{r(\cdot),\Omega} \leq c_0$, where $r := \min\{2,p\} \in C^{0,\alpha}(\overline{\Omega})$, then

$$\begin{aligned}\|\mathbf{F}_h(\cdot,\mathbf{D}\mathbf{v}_h)-\mathbf{F}_h(\cdot,\mathbf{D}\mathbf{v})\|_{2,\Omega}^2+\|q_h-q\|_{p'(\cdot),\Omega}^{(r^-)'} &\lesssim h^{2\alpha}\,\|1+|\mathbf{D}\mathbf{v}|^{p(\cdot)s}\|_{1,\Omega}\\ &\quad+h^{2\beta}\,[\mathbf{F}(\cdot,\mathbf{D}\mathbf{v})]_{N^{\beta,2}(\Omega)}^2\\ &\quad+h^{\min\{2,(p^+)'\}\gamma}\big(\|(\varphi_{|\mathbf{D}\mathbf{v}|})^*(|\nabla^\gamma q|)\|_{1,\Omega}+\||\nabla^\gamma q|\|_{p'(\cdot),\Omega}^{(r^-)'}\big)\,.\end{aligned}\tag{3}$$

The proof of the *a priori* error estimate (3) resorts to several fractional interpolation error estimates, including special fractional interpolation error estimates tailored to the convective term. More precisely, the latter was newly established in [28, Sec. 5]. Apart from that, different from initially expected, it was possible to derive the *a priori* error estimate (3) without imposing any further regularity of the pressure than the natural regularity one can usually expect as above.

Objective A2.2: An objective of this work package is to derive *a priori* error estimates for a finite element approximation of the steady $p(x)$-Navier–Stokes equations (similar to (3)) in which the derived error decay rates do not depend critically on the maximal and the minimal value of the power-law index, *i.e.*, on $p^-, p^+ \in (1,\infty)$.

Result A2.2: If $p \in C^{0,\alpha}(\overline{\Omega})$, $\alpha \in (0,1]$, with $p^- \geq 2$, $\delta > 0$, $\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) \in N^{\beta,2}(\Omega;\mathbb{R}^{d\times d})$, $\beta \in (0,1]$, and $q \in H^{\gamma,p'(\cdot)}(\Omega)$, $\gamma \in (\frac{\alpha}{\min\{2,(p^+)'\}},1]$, and $(\delta+|\mathbf{D}\mathbf{v}|)^{\frac{p(\cdot)-2}{2}}|\nabla^\gamma q| \in L^2(\Omega)$, it was established (*cf.* [28, Cor. 6.2(6.4)]) that there exists a constant $s>1$, which can chosen to be close to 1 if $h>0$ is close to 0, and a constant $c_0>0$ such that if $\|\mathbf{D}\mathbf{v}\|_{r(\cdot),\Omega} \leq c_0$, where $r := \min\{2,p\} \in C^{0,\alpha}(\overline{\Omega})$, then

$$\begin{aligned}\|\mathbf{F}_h(\cdot,\mathbf{D}\mathbf{v}_h)-\mathbf{F}_h(\cdot,\mathbf{D}\mathbf{v})\|_{2,\Omega}^2+\|q_h-q\|_{p'(\cdot),\Omega}^{(r^-)'} &\lesssim h^{2\alpha}\,\|1+|\mathbf{D}\mathbf{v}|^{p(\cdot)s}\|_{1,\Omega}\\ &\quad + h^{2\beta}\,[\mathbf{F}(\cdot,\mathbf{D}\mathbf{v})]_{N^{\beta,2}(\Omega)}^2\\ &\quad + h^{2\gamma}\left(\|(\delta+|\mathbf{D}\mathbf{v}|)^{\frac{2-p(\cdot)}{2}}|\nabla^\gamma q|\|_{2,\Omega}^2+\||\nabla^\gamma q|\|_{p'(\cdot),\Omega}^2\right).\end{aligned} \tag{4}$$

In the case $p^- \geq 2$, $\mathbf{f} \in L^2(\Omega;\mathbb{R}^d)$, and $\alpha=\beta=\gamma=1$, it has been established (*cf.* [28, Lem. 3.5(ii)]) that from $\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) \in W^{1,2}(\Omega;\mathbb{R}^{d\times d})$, it follows that $(\delta+|\mathbf{D}\mathbf{v}|)^{\frac{p(\cdot)-2}{2}}|\nabla q| \in L^2(\Omega)$, so that, in this case, the additional regularity assumption on the kinematic pressure is reasonable. A similar statement for the case $\alpha \in (0,1)$ could not yet be established (due to technical tools yet missing, the derivation of which would have gone beyond the scope of this research project), but remains as a conjecture.

Work Package A3: Experimental investigation of a finite element (FE) approximation.

Objective A3.1: An objective of this work package is to investigate the influence of the α-Hölder regularity of the power-law index $p \in C^{0,\alpha}(\overline{\Omega})$, $\alpha \in (0,1]$, on the error decay rates *via* numerical experiments.

Result A3.1: According to [20, Rem. 4.5], if $p \in C^{0,\alpha}(\overline{\Omega})$, $\alpha \in (0,1]$, one can only hope for the *"partial"* regularity $\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) \in N^{\alpha,2}(\Omega;\mathbb{R}^{d\times d})$. In the case $p^- \geq 2$ and $\alpha=1$, it has been established (*cf.* [28, Lem. 3.5(i)]) that from $\mathbf{F}(\cdot,\mathbf{D}\mathbf{v}) \in W^{1,2}(\Omega;\mathbb{R}^{d\times d})$, it follows that $q \in W^{1,p'(\cdot)}(\Omega)$. A similar statement for the case $\alpha \in (0,1)$ could not yet been established (due to technical tools yet missing, the derivation of which would have gone beyond the scope of this research project), but remains as a conjecture. Thus, in the *a priori* error estimates (3), (4), especially the case $\alpha=\beta=\gamma$ is relevant and, in this case, the expected error decay rate depending on $\alpha \in (0,1]$ could be measured in numerical experiments (*cf.* Result A3.2 and [28, Subsec. 7.2]). This dependence, however, may be traced back to the reduced regularity of the velocity vector field and the kinematic pressure as, in the case $\alpha \in (0,1)$ and $\beta=\gamma=1$, the expected error decay rate depending on $\alpha \in (0,1]$ could not be measured yet (*cf.* Figure 1).

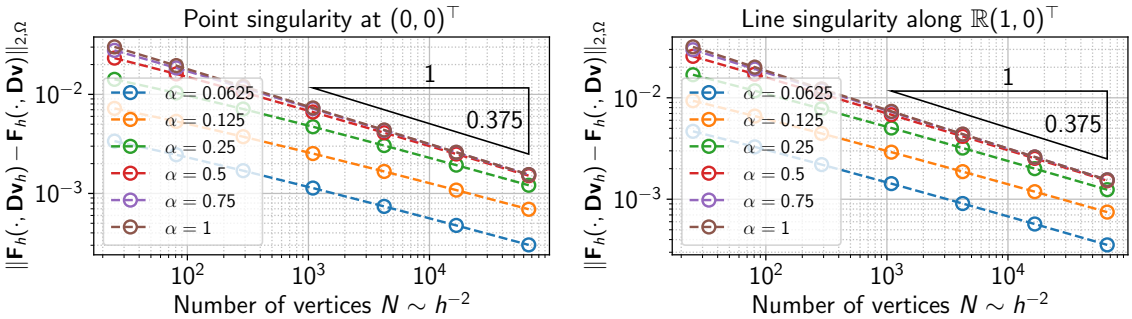

Figure 1: Error decay for the same manufactured solution as in [28, Subsec. 7.2, (Case 1)], in the case $\alpha \in \{0.0625, 0.125, 0.25, 0.5, 0.75, 1.0\}$ and $\beta = \gamma = 1$: *left*: the power-lax index has a point singularity at $(0.0)^\top$; *right*: the power-lax index has a line singularity along $\mathbb{R}(1.0)^\top$.

Objective A3.2: An objective of this work package is to confirm the quasi-optimality of the *a priori* error estimates (3), (4) derived in Work Package A3.1 *via* numerical experiments.

Result A3.2: The quasi-optimality of the *a priori* error estimates for the velocity vector field (3), (4) derived in Work Package A3.1 was confirmed *via* numerical experiments (*cf.* [28, Subsec. 7.2]). However, the quasi-optimality of the *a priori* error estimates (3), (4) for the kinematic pressure derived in Work Package A3.1 could not yet be confirmed (*cf.* [28, Subsec. 7.2]).

Objective: A3.3: An objective of this work package is to investigate the influence of the choice of quadrature points employed to discretize the power-law index in a finite element approximation of the steady $p(x)$-Navier–Stokes equations on the error decay rates *via* numerical experiments.

Result A3.3: For the same manufactured solution as in [28, Subsec. 7.2, (Case 1)], numerical experiment were carried out in which the power-law index is approximated either using nodal interpolation into globally continuous element-wise polynomial functions or using a (local) L^2-projection operator onto element-wise polynomial functions, each of polynomial degrees $p_{\mathrm{deg}} \in \{1, \ldots, 6\}$. In each case, no improvement compared to [28, Subsec. 7.2, (Case 1)] could be reported, neither in terms of the smallness of error nor in terms of the asymptotic behaviour of the error decay (*cf.* Figure 2).

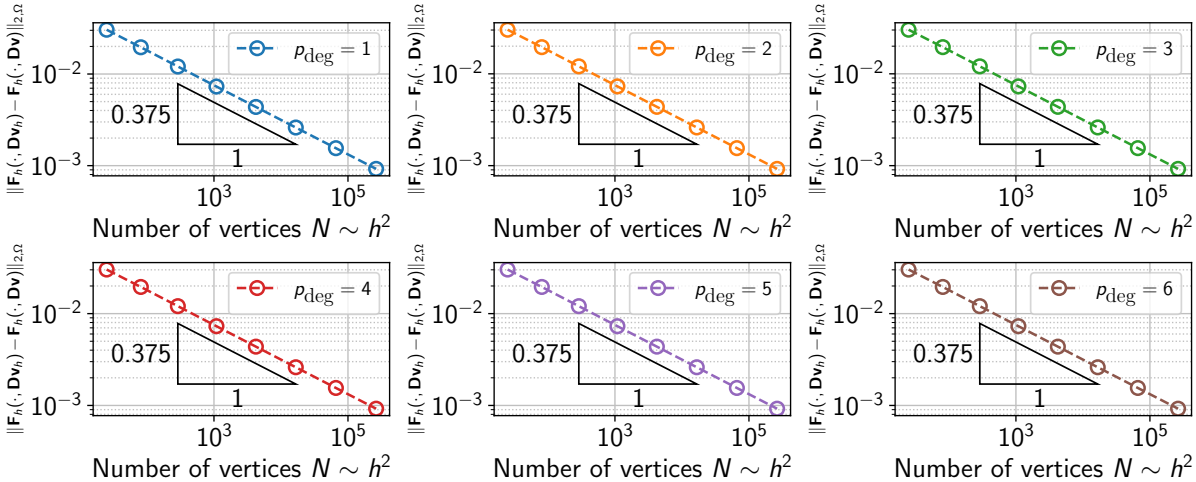

Figure 2: Error decay for the same manufactured solution as in [28, Subsec. 7.2, (Case 1)] when the power-law index is approximated using nodal interpolation into globally continuous element-wise polynomial functions of polynomial degrees $p_{\mathrm{deg}} \in \{1, \ldots, 6\}$. The same results were reported when the power-law index is approximated using a (local) L^2-projection operator onto element-wise polynomial functions of polynomial degrees $p_{\mathrm{deg}} \in \{1, \ldots, 6\}$.

Work Area B: FE approximation of the unsteady $p(t,x)$-Navier–Stokes equations.

This work area aims at theoretical and experimental investigations on a fully-discrete Rothe–Galerkin approximation of the unsteady $p(t,x)$-Navier–Stokes equations (1):

Work Package B1: Weak convergence of a fully-discrete Rothe–Galerkin approximation.

Objective B1: The objective of this work package is to establish a general framework that allows to conclude the well-posedness (*i.e.*, existence of iterates), stability (*i.e.*, *a priori* estimates), and weak convergence of a fully-discrete Rothe–Galerkin approximation of an unsteady problem, containing the unsteady $p(t,x)$-Navier–Stokes equations (1) as a particular case, for a $\log$-Hölder continuous power-law index $p\colon Q_T \to (1,+\infty)$ with $p^- \geq 2$ (or $p^- \geq \frac{3d+2}{d+2}$ if the convective term is present) and general approximations $(p_h^\tau)_{\tau,h>0} \subseteq L^\infty(Q_T)$ satisfying $p_h^\tau \to p$ $(\tau, h \to 0)$.

Result B1: On the basis of the concept of the non-conforming Bochner condition (M) (*cf.* [29, Sec. 6]), a general framework (*cf.* [29, Sec. 7]) was established that allows to conclude the well-posedness, stability, and weak convergence of a fully-discrete Rothe–Galerkin approximation for a $\log$-Hölder continuous power-law index $p\colon Q_T \to (1,+\infty)$ with $p^- \geq 2$ (or $p^- \geq \frac{3d+2}{d+2}$ if the convective term is present) and general approximations $(p_h^\tau)_{\tau,h>0} \subseteq L^\infty(Q_T)$ satisfying $p_h^\tau \to p$ $(\tau, h \to 0)$. This framework applies to a large class of unsteady problems, including, *e.g.*, the unsteady $p(t,x)$-Stokes equations (*cf.* [29, Subsec. 8.4]) and the unsteady $p(t,x)$-Navier–Stokes equations (*cf.* [29, Subsec. 8.5]).

Work Package B2: Experimental investigations on a fully-discrete Rothe–Galerkin approximation.

Objective B2: The objective of this work package is to confirm experimentally the weak convergence of a fully-discrete Rothe–Galerkin approximation of the unsteady $p(t,x)$-Navier–Stokes equations (1), in the case of low regularity data.

Result B2: Since the weak convergence of a fully-discrete Rothe–Galerkin approximation is difficult to measure in numerical experiments, manufactured solutions with fractional regularity properties were constructed, the fractional regularity parameters were gradually reduced, and error decay rates that reduce with decreasing (but are stable for fixed) value of the fractional regularity parameter were measured (*cf.* [29, Subsec. 9.3]). This gave an indication that weak convergence is –at least– likely.

Work Package B3: Error estimates for a fully-discrete Rothe–Galerkin approximation.

Objective B3: The objective of this work package is to derive *a priori* error estimates (similar to (3), (4)) for a fully-discrete Rothe–Galerkin approximation of the unsteady $p(t,x)$-Navier–Stokes equations (1), using Témam's modification (*cf.* [24]) to discretize the convective term and a one-point quadrature rule to discretize the power-law index, imposing only fractional regularity assumptions on the velocity vector field and the kinematic pressure (similar to the Objectives A2.1 and A2.2 or the contributions [23, 28]).

Result B3: A priori error estimates (similar to (3), (4)) for a fully-discrete Rothe–Galerkin approximation of the unsteady $p(t,x)$-Stokes equations, using a one-point quadrature rule to discretize the power-law index, imposing only fractional regularity assumptions on the velocity vector field and the pressure have been derived: more precisely, if $p \in C^{0,\alpha_{\mathrm{t}},\alpha_{\mathrm{x}}}(\overline{Q_T})$, $\alpha_{\mathrm{t}}, \alpha_{\mathrm{x}} \in (0,1]$, *i.e.*, for every $(t,x)^\top, (s,y)^\top \in Q_T$, we have that

$$|p(t,x) - p(s,y)| \lesssim |t-s|^{\alpha_{\mathrm{t}}} + |x-y|^{\alpha_{\mathrm{x}}}\,,$$

with $p^- > 1$,$\mathbf{v} \in L^{p(\cdot,\cdot)}(Q_T;\mathbb{R}^d) \cap L^\infty(I;N^{\delta_{\mathrm{x}},2}(\Omega;\mathbb{R}^d))$, $\delta_{\mathrm{x}} \in (0,1]$, with $\mathbf{v}(t) \in W_0^{1,p(t,\cdot)}(\Omega;\mathbb{R}^d)$ for a.e. $t \in I$, $\mathbf{F}(\cdot,\cdot,\mathbf{D}_x\mathbf{v}) \in N^{\beta_{\mathrm{t}},2}(I;(L^2(\Omega))^{d\times d}) \cap L^2(I;(N^{\beta_{\mathrm{x}},2}(\Omega))^{d\times d})$, $\beta_{\mathrm{t}} \in (\frac{1}{2},1]$, $\beta_{\mathrm{x}} \in (0,1]$, and $q \in L^{p'(\cdot,\cdot)}(Q_T)$ with $q(t) \in H^{\gamma_{\mathrm{x}},p'(t,\cdot)}(\Omega)$ for a.e. $t \in I$ and $|\nabla_x^{\gamma_{\mathrm{x}}} q| \in L^{p'(\cdot,\cdot)}(Q_T)$, $\gamma_{\mathrm{x}} \in (\frac{\alpha_{\mathrm{x}}}{\min\{2,(p^+)'\}},1]$, it was established (*cf.* [30]) that there exists a constant $s > 1$, which can chosen to be close to 1 if $\tau + h > 0$ is close to 0, such that

$$\begin{aligned}
\max_{k=0,\ldots,K} \|\mathbf{v}_h^k - \mathbf{v}(t_k)\|_{2,\Omega}^2 + \sum_{k=0}^{K} \tau\, \|\mathbf{F}_h(t_k,\cdot,\mathbf{D}_x\mathbf{v}_h^k) - \mathbf{F}_h(t_k,\cdot,\mathbf{D}_x\mathbf{v}(t_k))\|_{2,\Omega}^2 \\
\lesssim \tfrac{h^{2\delta_{\mathrm{x}}}}{\tau} [\mathbf{v}]_{L^\infty(I;N^{\delta_{\mathrm{x}},2}(\Omega))}^2 \\
+ (\tau^{2\alpha_{\mathrm{t}}} + h^{2\alpha_{\mathrm{x}}})\, \sup_{t\in I} \|1 + |\mathbf{D}_x\mathbf{v}|^{p(t,\cdot)s}\|_{1,\Omega} \\
+ \tau^{2\beta_{\mathrm{t}}}\, [\mathbf{F}(\cdot,\cdot,\mathbf{D}_x\mathbf{v})]_{N^{\beta_{\mathrm{t}},2}(I;L^2(\Omega))}^2 \\
+ h^{2\beta_{\mathrm{x}}}\, [\mathbf{F}(\cdot,\cdot,\mathbf{D}_x\mathbf{v})]_{L^2(I;N^{\beta_{\mathrm{x}},2}(\Omega))}^2 \\
+ \|(\varphi_{|\mathbf{D}_x\mathbf{v}|})^*(h_x^{\gamma_{\mathrm{x}}}|\nabla^{\gamma_{\mathrm{x}}} q|)\|_{1,Q_T}\,.
\end{aligned} \tag{5}$$

In the *a priori* error estimate (5), by $(\mathbf{v}_h^k)_{k=0,\ldots,K}$, $K \in \mathbb{N}$, we denote the iterates of a Rothe–Galerkin approximation of the unsteady $p(t,x)$-Stokes equations employing a backward Euler step in time and discretely inf-sup stable finite elements in space with respect to discretizations $\{\mathcal{Q}_h^\tau := \mathcal{I}_\tau \times \mathcal{T}_h\}_{\tau,h>0}$ of the time-space cylinder Q_T. Here, $\{\mathcal{I}_\tau := \{(t_{k-1},t_k)\}_{k=1,\ldots,K}\}_{\tau>0}$ with $t_k = k\tau$ for all $k = 0,\ldots,K$, *i.e.*, $\tau = \frac{T}{K}$, and $\{\mathcal{T}_h\}_{h>0}$ a family of shape-regular and conforming triangulations of $\overline{\Omega}$. Moreover, the non-linear mapping $\mathbf{F}_h\colon Q_T \times \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$, for every $t \in (t_{k-1},t_k)$, $k = 1,\ldots,K$, $x \in T$, $T \in \mathcal{T}_h$, and $\mathbf{A} \in \mathbb{R}^{d\times d}$, is defined by

$$\mathbf{F}_h(t,x,\mathbf{A}) := \mathbf{F}(t_k,\xi_T,\mathbf{A})\,,$$

where the point $\xi_T \in T$, for every $T \in \mathcal{T}_h$, is an arbitrary quadrature point and the non-linear mapping $\mathbf{F}\colon Q_T \times \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$, for every $(t,x)^\top \in Q_T$ and $\mathbf{A} \in \mathbb{R}^{d\times d}$, is defined by

$$\mathbf{F}(t,x,\mathbf{A}) := (\delta + |\mathbf{A}^{\mathrm{sym}}|)^{\frac{p(t,x)-2}{2}} \mathbf{A}^{\mathrm{sym}}\,.$$

3.2 Description of the handling of research data generated in the project and any data infrastructures used

The resulting data from the numerical experiments were published in preliminary publications (*cf.* [28, 29]). To create the experiments the open source software package `fenics` has been used. The program code used will be made publicly available after acceptance of the articles in scientific journals.

3.3 Bibliography

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4 Published project results

4.1 Peer-reviewed publications

None.

4.2 Further publications and published results

The following two pre-prints contain project results and are currently under review or in preparation:

- L. C. Berselli and A. Kaltenbach. *Error analysis for a finite element approximation of the steady $p(\cdot)$-Navier–Stokes equations*, submitted, 2023.
  URL: https://arxiv.org/abs/2311.00534
- L. C. Berselli and A. Kaltenbach. *Convergence analysis of a fully-discrete finite element approximation of the unsteady $p(\cdot,\cdot)$-Navier–Stokes equations*, submitted, 2024.
  URL: https://arxiv.org/abs/2402.16606
- L. C. Berselli, A. Kaltenbach, and S. Ko. *Error analysis for a fully-discrete finite element approximation of the unsteady $p(\cdot,\cdot)$-Stokes equations*, in preparation, 2024.

4.3 Patents (applied for and granted)

None.